Catching a fast robber on interval graphs

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Abstract. We analyse the Cops and ∞-fast Robber game on the class of interval graphs and prove it to be polynomially decidable on such graphs. This solves an open problem posed in paper "Pursuing a fast robber on a graph" by Fomin et al. [4] The game is known to be already NP-hard on chordal graphs and split-graphs.

The game is played by two players, one controlling k cops, the other a robber. The players alternate in turns, all the cops move at once to distance at most one, the robber moves along any cop-free path. Cops win by capturing the robber, the robber by avoiding capture.

The analysis relies on the properties of an interval representation of the graph. We show that while the game-state graph is generally exponential, every cops' move can be decomposed into simple moves of three types, and the states are reduced to those defined by certain cuts of the interval representation. This gives a restricted game equivalent to the original one together with a winning strategy computable in polynomial time.

Keywords: cop and robber game; pursuit game; combinatorial game; interval graph; interval graph representation

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1 Introduction

The recent development in the area of combinatorial "Cop and Robber" games (also called pursuit-evasion games) includes results on games with various speeds of the players and their computational complexity.

The Cops and s-fast Robber game is a generalisation of the original Cops and Robber game introduced by Nowakowski and Winkler [7] and by Quilliot [8] allowing the Robber to make s steps instead of 1.

In their paper "Pursuing a fast robber on a graph" [4], Fomin et al. propose the complexity of the ∞-fast Robber game on interval graphs as an open question which we answer. The proof is constructive and shows how to decompose seemingly complicated moves of the cops into simple basic moves.

Founding the boundary graph classes and restrictions for which the games are NP-hard or polynomially decidable shows which are the aspects of the problems that make it easy or hard, providing further insight into the problem.

In this paper we draw the line between interval graphs and split and chordal graphs. Our result is not only an example of a game reduction technique, but also indicates that the hardness of the problem on chordal graphs lies in their unbounded asteroidal number (and not in i.e. unbounded clique size or diameter).

See Section 7 for further discussion.

The **Cops and s-fast Robber game** is defined as follows: The game is played by k cops controlled by one player and one robber controlled by the second player on a given simple undirected graph G. The cops and the robber are positioned on vertices of G at all times, more cops may share a vertex. Both players have a complete information about G and the game state.

First, the cops choose starting vertices, then the robber chooses a starting vertex. One turn then consists of each of the cops moving to distance at most 1, and then by the robber moving along a cop-free path of length at most s. The robber may never move through a vertex occupied by a cop. Note that this restricts the robber to one component of the cop-free subgraph of G.

Should a cop be at or move to the robber's vertex, the cops immediately win. The robber wins by avoiding the capture indefinitely.

This game is equivalent to the original Cop and Robber game for $s = 1$.

Some of the complexity aspects of Cops and s-fast Robber games are examined by Fomin et al. [5, 4]. They show that for all s, these games are NP-hard (and even W[2]-hard in the version parametrised by k) to decide even on chordal graphs while still being polynomially decidable on interval graphs.

The Cops and ∞-fast Robber game is the limit of the sequence of Cop and s-fast Robber game, but is an interesting game on its own. The hardness results of Fomin et al. easily extend to this game but the polynomiality proofs fail.

One of the main ideas of this paper is that any cops' winning strategy on an interval graph *essentially* consists only of *sweeping* and *splitting* moves on the interval representation of the graph. We use a formal version of this statement to show that the game can be decided in polynomial time.

Theorem 1. *There is a polynomial-time algorithm deciding a Cop and ∞-fast Robber game with k cops on a given graph G.*

2 Preliminaries

In this paper, we use "the" standard graph- and game-theoretic notation. For introduction to these areas, we recommend the books Modern Graph Theory [2] and Lessons in Play: An Introduction to Combinatorial Game Theory [1].

We use $N(v)$ and $N[V]$ to denote the closed neighbourhood (including v, resp. V) of a vertex or a set of vertices, respectively. We briefly mention some of the less-known graph classes and their properties:

A graph G is *chordal* (also called *bridged*) if there are no induced cycles of length at least 4. A graph is a *split graph* if its vertices can be partitioned into two sets I and K, I an independent set and K a complete subgraph. Every split graph is also chordal.

A graph is an *interval graph* if it can be realised as an intersection graph of a family of intervals on a real line. For a family of intervals $\mathcal{I}$, the associated intersection graph $G(\mathcal{I})$ has one vertex for each of the intervals and an edge between the vertices corresponding to intervals I_1 and I_2 if and only if $I_1 \cap I_2 \neq \emptyset$. Every interval graph is chordal.

Throughout the article we assume that every interval graph comes with a fixed interval representation consisting of *open* intervals with integral endpoints $1, 2, \dots 2|V_G|$ and such that no two intervals share an endpoint. Note that such an interval representation can be reconstructed from G in linear time, as shown by Korte and Mhring [6].

We identify the intervals of the chosen representation with the vertices, interchanging these frequently.

In the rest of the paper, we fix G to be a connected interval graph with a given interval representation, V be the set of vertices of G and E be the set of its edges. We also fix k to be the given number of cops.

Note that in a disconnected graph, the cops have to decide on a distribution among the components and the robber then chooses a component to play in. Later in game only the cops in that component matter. Assuming we can compute the minimum number of cops necessary for each separate connected component in polynomial time, we can just sum up these numbers to get the minimum number of cops that can capture the robber in the entire graph.

We slightly modify the game: all the cops must start at the leftmost interval of G. This is equivalent to the game defined above, since the cops can first move to any desired position (ignoring the robber) and then pretend that the game started from that position.

Formally we call the game state before cops' move $\mathcal{C}(C, r)$, where C is a multiset of vertices occupied by the cops and r is the vertex the robber moved to. The game state before robber's move is $\mathcal{R}(C, A)$ with C as above and A is the set of all vertices the robber may move to (the connected component of $G - C$ containing the robber).

This slightly reduces the complexity of the examined states. Note that before any robber's turn, two states with robber in the same component of $G - C$ offer the same moves to the robber. Also note that A is always connected including the first round.

The initial state of the game is $\mathcal{R}(C, V - C)$, where C is the multiset containing k times the leftmost vertex of G. We also add a special state $\mathcal{WIN}$, which denotes the capture of the robber.

3 Specifics of the game

In this section, we introduce the definitions necessary for the game strategy reduction. These allow a more precise look at the game.

The main idea of this paper is to *simulate* arbitrarily complex cops' winning strategy by a simplified, *restricted* strategy using only three kinds of actions – *sweep*, *split* and *endgame*. The definitions of the actions are provided below together with a formal definition of a restricted strategy.

Informally, an action is a predefined part of cops' strategy starting at a distinguished game state and ending at another distinguished game state. A restricted cops' strategy may only decide the next action in such distinguished game state.

Before defining the restricted actions, let us introduce some useful concepts and make few observations.

Proposition 1. *At any point in the game, the cops restrict robber's movement to an interval, the robber may move to arbitrary vertex of this interval except those occupied by the cops.*

The union of vertices (as intervals) reachable by the robber is obviously a single interval (this interval does not generally correspond to a vertex). We call this interval the *playground*. Any cop-free vertex inside or incident with the playground is adjacent to a vertex accessible to the robber.

The left and right endpoints of the playground are called the left and right *barriers*. We denote the playground with barriers L and R, $L < R$ by (L, R). Note that L and R are not contained in the playground (as all the intervals are open) and all intervals incident with these barriers are be occupied by the cops (otherwise the playground would be bigger).

The vertices incident to the barrier are called the *barrier's support*, denoted $V(L)$. Note that the support of either barrier may be empty. The vertices contained *inside* a playground (L, R) are called the *playground support*, denoted $V(L, R)$. Support of a playground is always disjoint from the supports of the barriers.

From the cops occupying a barrier support, choose and fix one cop per vertex. Let us call these cops the cops *holding* the barrier. The choice and fixing of the cops is is mostly symbolic since the cops are indistinguishable, but is useful in some proofs. Note that a cop may hold both barriers at once, but as we will see, that this may happen only just before capturing the robber.

A playground (L, R) is *feasible*, if $|V(L) \cup V(R)| \leq k$, that is if the cops are able to hold both barriers at once. For every *nontrivial* feasible playground (L, R) (nonempty and not containing all the vertices), we fix a *canonical* game state

$$\S(L, R) = \mathcal{R}(V(L) \cup V(R), V(L, R))$$

with all extra cops not explicitly holding any vertices positioned at the leftmost vertex outside $V(L, R)$. A game state corresponding to an empty playground would be $\mathcal{WIN}$, as the cops must have just caught the robber.

There is a minor technicality – there is no game state corresponding to the *full* playground containing all the vertices, as there is no way to place the cops and get the playground to be the entire graph. We fix the state canonical for $(-\infty, \infty)$ to be the state $\mathcal{R}(\{v_0\}, V - v_0)$ with v_0 the left-most vertex. This makes the playground $(-\infty, \infty)$ always feasible.

Proposition 2. *If* (L, R) *is a feasible playground then the playground corresponding to its canonical game state is the same, or a subset in case of* (L, R) *full playground. As the full playground occurs only in the first move, we usually omit that case.*

The cops in position C *threaten* to take a vertex set B if the cops can occupy every vertex of B after one move. This is equivalent to an existence of a matching of C (as a multiset) with all vertices of B.

In our strategy, B is usually a barrier and there may be additional explicit conditions – usually, some other vertices must be held both before and after the move. When considering a set of cops threatening B, we fix a matching between the threatening cops and the vertices of B for the moment.

Now we are ready to define cops' restricted strategy. While a general cops' strategy is mapping from *every* valid state of the game to a move valid in that state, a *restricted cop's strategy* is a mapping from the robber states canonical to some feasible playground to actions valid in that state. Therefore, the restricted strategy can influence the game only after an action has finished (which it always does either in a state canonical to some playground or in $\mathcal{WIN}$).

4 Essential cops' moves

Formally, a *restricted cops' strategy* is a cops' strategy that, in a canonical game state, may only choose one valid action or pass. The chosen action is then played as a sub-strategy. After the action finishes, the strategy may choose another action.

An *action* is a sequence of moves starting at a canonical state and ending in one of defined canonical states or in $\mathcal{WIN}$. An action is *valid* if it can be played out by k cops.

The actions are ordered by preference. If an *endgame* move is valid from some playground, then we do not consider the possibility of other moves. Also, if a *sweep* from (L, R) to (L', R) is valid, we consider no *splits* to (L', R) or (L, R') for any R', and symmetrically for *sweep* to (L, R'). This saves us few cases in the technical parts of the proofs.

Endgame from $\S(L, R)$. From the state $\S(L, R)$, the cops assume a position C holding both L and R. C dominates $V(L, R)$. In the next move, the cops capture the robber. Such a multiset C with $|C| \leq k$ is a *witness* of the validity of the action.

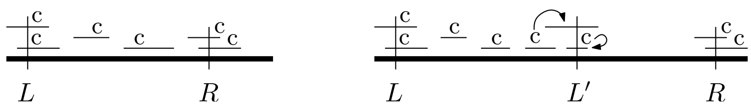

Fig. 1. Illustrations of *endgame* and *sweep* actions.

Sweep from $\S(L,R)$ **to** $\S(L',R)$ (or to $\S(L,R')$). From the state $\S(L,R)$, the cops assume a position C holding L and R. In C they threaten barrier L' (or R') while holding R (or L). If $L < L'$ (or $R' < R$) then C also dominates $V(L,L')$ (or $V(R',R)$). Then the cops in one move stop holding L (or R) and either capture the robber or take and start holding L' (or R') and move to the state canonical for the goal state.

The action is valid if such set C with $|C| \le k$ exists and an *endgame* from (L,R) is not possible. After the action, the robber is either in the new playground or captured, as (L,L') (or (R',R)) was dominated.

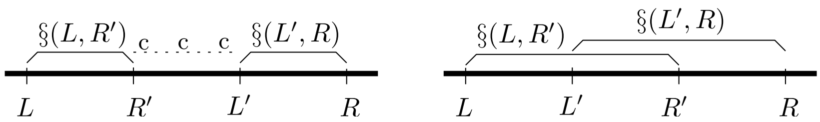

Fig. 2. Split action with $R' < L'$ and $R' \ge L'$.

Split from $\S(L,R)$ **to either** $\S(L',R)$ **or to** $\S(L,R')$**.** From the state $\S(L,R)$, the cops assume a position C holding both L and R. In C they threaten barrier L' while holding R *and* threaten R' while holding L. If $R' < L'$ then C also dominates $V(R',L')$.

After moving to C, the cops let the robber move and decide according to robber's position r. If r is adjacent to a cop, they capture the robber. If $r \in V(L,R')$, the cops stop holding R and take and start holding R'. Otherwise, the cops stop holding L and take and start holding L'. If the cops did not capture the robber, they then assume the state canonical for either $\S(L,R')$ or $\S(L',R)$.

The action is valid only if such witness set C with $|C| \le k$ exists and neither *endgame* from $\S(L,R)$ nor *sweep* from $\S(L,R)$ to one of $\S(L,R')$, $\S(L',R)$ is possible. This last condition is added only for simplicity and could be dropped.

The existence of a witness of any of the action is decidable in polynomial time. The proof of the lemma below is somewhat technical and we present it to Section 6. The algorithms are mostly straightforward modifications of the well-known greedy minimum dominating set algorithms for interval graphs.

Lemma 1. *There are polynomial-time algorithms deciding the validity of actions endgame, sweep and split.*

This allows us to decide the existence of a cops' restricted strategy.

Theorem 2. *There is a polynomial time algorithm that, given an interval graph G and k, decides the existence of a winning restricted strategy for k cops.*

The length of a winning strategy (measured in game turns) is $O(|V_G|^3)$.

Proof. We construct a game-state digraph D representing the game with cops using restricted strategy.

The *cop-states* of D are all feasible playgrounds and $\mathcal{WIN}$. The empty playground is the initial cop-state, $\mathcal{WIN}$ is the cop-win state.

The *robber-states* of D are quadruples (L, L', R', R) of barriers such that *split* from (L, R) to either (L, R') or (L', R) is a valid action.

For every valid *endgame* from $\S(L, R)$, there is a move from (L, R) to $\mathcal{WIN}$. For every valid *sweep* from $\S(L, R)$ to $\S(L', R')$, there is a cop-move from (L, R) to (L', R'). For every valid *split* from $\S(L, R)$ to either $\S(L, R')$ or $\S(L', R)$, there is a cop-move from (L, R) to (L, R, L', R') and two robber-moves from (L, R, L', R') to (L, R') and (L', R).

We decide the game given by D using a general state-marking algorithm, giving us either a winning strategy for the cops or a non-losing strategy for the robber.

If we get a cops' winning strategy in D, it is also a restricted cops' strategy for the general game, where the cops play the original action of the move dictated by the strategy.

On the other hand, the moves of any restricted cops' strategy are present in the game-state digraph. Both cops' restricted strategy and the game encoded by D ignore the position of the robber in the playground except for the final move in *split* (that corresponds to the robber choosing one of the playgrounds). The game of D does not allow the robber to get captured prematurely, but that possibility is ruled out by assuming the robber to play optimally.

The game digraph D has $O(n^4)$ states and moves. The feasibility of every state and the validity of every move can be decided in polynomial time according to Lemma 1. The general combinatorial game-decision algorithm then runs in time polynomial in the size of D.

To see the maximum length of a strategy, note that an optimal cops' strategy visits every state at most once, there are $O(|V_G|^2)$ cop-states and every robber-state is followed by a cop-state. It is easy to see that playing out any single action takes $O(n)$ moves. □

5 Simulating general cops' strategy

In this section we prove the following theorem.

Theorem 3. *For an interval graph G and an integer k, k cops have a winning strategy for the Cops and ∞-fast Robber game if and only if k cops have a restricted winning strategy.*

With this, we may immediately prove Theorem 1.

Proof of Theorem 1. According to Theorem 2, we can decide the existence of a winning restricted cops' strategy in polynomial time. By Theorem 3, such strategy exists if and only if a general winning cops' strategy exists. □

Proof of Theorem 3. The "if" part is straightforward, as the cops can play out the actions of a restricted winning strategy. The action properties ensure that the actions are possible and that the robber is inside the respective playground or captured.

For the other direction, let $\mathcal{S}$ be an optimal (in the terms of length) cop's winning strategy for k cops. We may assume that the strategy wins from any state, as it may move the cops to the state canonical to the full playground and then play from there. Note that if the cops play according to $\mathcal{S}$, the game will never revisit a game state.

Let S be the subgraph of the game state digraph representing the strategy. The vertices are all the cop- and robber-states of the game. From each cop-vertex, there is exactly one cop-move as dictated by $\mathcal{S}$. From each robber-vertex, all the robber-moves are present. Note that S is acyclic.

Fix any total ordering o of the states of S extending the partial order given by the moves.

The restricted cops' winning strategy is obtained using the following lemma:

Lemma 2. *Given any robber state $\mathcal{R}(C, A)$ of S, let $\mathcal{R}(C_i, A_i)$ be the robber states of S reachable from $\mathcal{R}(C, A)$ in two moves (a robber move and a cop move). Let $P = \S(L, R)$ be a playground corresponding to $\mathcal{R}(C, A)$ and $P_i = \S(L_i, R_i)$ be the playgrounds corresponding to $\mathcal{R}(C_i, A_i)$.*

There is a restricted cops' strategy that starts in the cop-state canonical to $\S(L, R)$ and moves a cop-state canonical to one of $\S(L_i, R_i)$ in $O(n)$ actions, or wins.

The desired restricted strategy $\mathcal{T}$ is as follows:

For a cop-state canonical to $\S(L, R)$, find the latest (w.r.t. o) robber-state $\mathcal{R}(C, B)$ with playground $\S(L, R)$. Let $\mathcal{C}(C, r_i)$ be all the possible following cop-states and let $\mathcal{R}(C_i, R_i)$ and $\S(L_i, R_i)$ be the resulting states of the cop-moves dictated by $\mathcal{S}$ and the corresponding playgrounds.

The restricted strategy $\mathcal{T}$ should play the actions given by Lemma 2. This leaves the game in a state canonical to one of $\S(L_i, R_i)$ or won for the cops. Note that all $\S(L_i, R_i)$ are different from $\S(L, R)$, because the state $\mathcal{R}(C, B)$ is latest such state.

Now we have that the latest occurrence of a robber-state with playground $\S(L_i, R_i)$ is (w.t.r. o) larger than that of $\S(L, R)$. Therefore, by playing $\mathcal{T}$, the latest state (w.r.t. o) with the same playground as the current one only increases.

This proves that $\mathcal{T}$ is acyclic and therefore winning for the cops. □

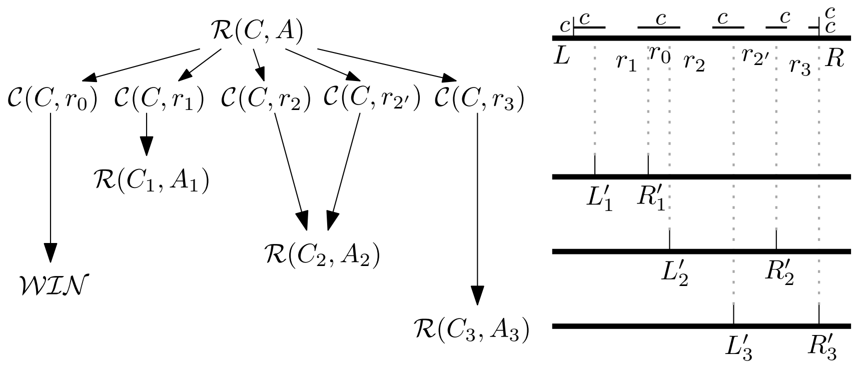

Fig. 3. An illustration of one robber state $\mathcal{R}(C, R)$ of S with some of the robber moves r_i followed by the cop moves dictated by S, and the corresponding playgrounds.

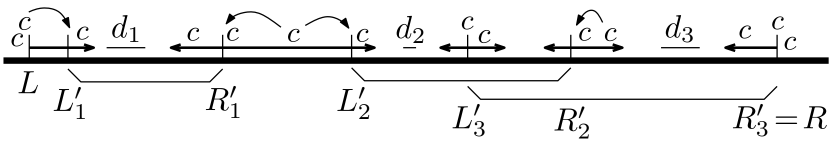

Fig. 4. An illustration of the playgrounds of $\mathcal{P}$. Note that the playgrounds may overlap. This also illustrates how the d_j separate the groups of cops of C.

Proof of Lemma 2. Let D be the non-dominated vertices of P, that is, $D = P \setminus N[C]$. If $D = \emptyset$, then C is a witness for *endgame* and the strategy may immediately win.

Otherwise, let $\mathcal{P}$ be an inclusion-minimal sub-collection of playgrounds covering D, indexed by $j \in \{1 \ldots\}$ from left to right. Denote these playgrounds $P'_j = \S(L'_j, R'_j)$ and the corresponding states $\mathcal{R}(C'_j, B'_j)$. For every P'_j, there is some vertex d_j of D covered only by P'_j, otherwise $\mathcal{P}$ is not minimal.

The cops of C threaten every individual barrier L'_j and R'_j. As every P'_j contains an non-dominated vertex, the cops threatening L'_j and R'_j are disjoint for every j.

The strategy construction is algorithmic and reduces $\mathcal{P}$ and modifies C on the way. However, multiset C is modified only the left of d_1 and to the right of $d_{|\mathcal{P}|}$. After the first two actions, the playground is always $\S(L'_1, R'_{|\mathcal{P}|})$.

The strategy starts with a *sweep* to $\S(L'_1, R)$ followed by a *sweep* from $\S(L'_1, R)$ to $\S(L'_1, R'_{|\mathcal{P}|})$. The first *sweep* is valid as the cops of C to the left of d_1 together with $V(R)$ witness. The validity of the second *sweep* is witnessed by the cops of C to the right of $d_{|\mathcal{P}|}$ together with $V(L'_1)$. This changes the position

If there is only one playground $\S(L'_1, R'_1)$ in $\mathcal{P}$, we have that the current playground is $\S(L'_1, R'_1)$ and we are done.

If there are multiple playgrounds in $\mathcal{P}$, let the strategy play a *split* to either $\S(L'_1, R'_1)$ or $\S(L'_2, R'_{|\mathcal{P}|})$. The *split* is witnessed by C, as the original C threatened both $V(R'_1)$ and $V(L'_2)$, dominating $V(R'_1, L'_2)$ if $R'_1 < L'_2$. Note that the cops of C between d_1 and d_2 are unmodified.

This either moves the game to state canonical to $\S(L'_1, R'_1)$, or reduces $\mathcal{P}$. Multiset C is modified by taking $V(L'_2)$, but (as this was possible in the original C) this may be done only using the cops to the left of d_2 and right of d_1. In every step, we renumber the elements of $\mathcal{P}$ from 1 up.

Repeating this step, the strategy either wins or moves to the state canonical to some $\S(L'_j, R'_j)$. ◻

6 Complexity of finding witness

In this section we prove the complexity of the three decision problems in Lemma 5, splitting it into the following three lemmas. The statement of Lemma 5 then directly follows.

The algorithms are not given explicitly, but follow straightforwardly by following the order of assumptions in the proofs of the lemmas.

Lemma 3. *A smallest witness for endgame from $\S(L, R)$ can be computed in polynomial time.*

Proof. Both $V(L)$ and $V(R)$ have to be occupied and $D := V(L, R) - N[V(L) \cup V(R)]$ has to be dominated (note that all the vertices of G can be used for the domination). The size of a smallest witness is then $|V(L) \cup V(R)| + \mathrm{dom}_G(D)$.

The size of $\mathrm{dom}_G(D)$ is computed by a greedy sweep algorithm for a minimum dominating set in interval graphs. See a paper by Brandstädt [3] for details. ◻

Lemma 4. *A smallest witness for sweeps from $\S(L, R)$ to $\S(L', R)$ and from $\S(L, R)$ to $\S(L, R')$ can be computed in polynomial time.*

Proof. We prove only the direction from $\S(L, R)$ to $\S(L', R)$ as the other is symmetrical.

Let C be a smallest witness and M a matching of every cop of C with the threatened vertex such that C holds R (at every vertex of $V(R)$ there is a loop in the matching). Note that C occupies L.

We can ensure that M uses the maximum number of cops from $V(L)$ just by changing M. Let the vertices threatened by cops from $V(L)$ be $L'_C \subseteq V(L')$. We fix this part of M. M now threatens $L'' := V(L') - L'_C$ without cops at $V(L)$ or $V(R)$ (and optionally dominates $V(L, L')$).

If $L' < L$, then C has to contain $|L''|$ additional cops threatening L''. These cops can be positioned at the threatened vertices and altogether, $|V(L) \cup V(R)| + |L''|$ cops are required.

If $L' > L$, let $D := V(L, L') - N[V(L) \cup V(R)]$ and let $c_1, c_2, \ldots$ be the cops dominating D ordered left to right. We can ensure that every cop c_i is on the rightmost-ending vertex such that D is still dominated, since moving c_i to such vertex v preserves both domination of D and (possible) threatening of some vertex $l \in L''$. For the latter note that v lies either left of L or $v \in V(L)$. Note that such c_i are exactly those returned by a left-to-right sweep greedy algorithm computing $\mathrm{dom}_G(D)$.

Now we may assume that M is maximal between L'' and c_i's. Therefore, C must contain at least one extra cop for every $v \in L''$ not matched to some c_i. This gives the total size of C. □

Lemma 5. *A smallest witness for split from $\S(L, R)$ to either $\S(L', R)$ or to $\S(L, R')$ can be found in polynomial time.*

Proof. Let C be a smallest witness and M_L (resp. M_R) be matchings of cops of C with $V(L') \cup V(R)$ (resp. $V(R') \cup V(L)$) such that C holds R in M_L (resp. C holds L in M_R) as in the proof of Lemma 4.

We can ensure that both M_L and M_R use the maximum number of cops from $V(L)$ and $V(R)$ just by changing M. Let the vertices of $V(L')$ threatened by cops from $V(L)$ be L'_C and symmetrically, let R'_C be the vertices of $V(R')$ threatened by cops from $V(R)$. We fix these parts of M_L and M_R. Let $L'' := V(L') - L'_C$ and $R'' := V(R') - R'_C$. C now threatens both L'' (in M_L) and R'' (in M_R) without the cops at $V(L)$ and $V(R)$ (resp. those occupying/holding these barriers) and optionally dominates $V(R', L')$.

Case 0. If there is an interval (vertex) v of G containing both R' and L', then the cops of C threatening L'' and R'' can be assumed to be positioned at v. This also makes $V(R', L')$ dominated. In this case we can take $C = V(L) \cup V(R) \cup (\{v\} \times \max(|L''|, |R''|))$.

In the remaining cases we assume that there is no such vertex v.

Case $L' < R'$. For every $l \in L''$, we may assume that the cop c threatening l in M_L is positioned on the neighbour v of l with the rightmost right end. If cop c also threatens some vertex $r \in R''$ then we still have $v \in N[r]$. Note that if v would lie to the right of R' then vertex l would satisfy conditions of Case 0. Let T_L be the cops threatening L'' in M_L.

We can make M_R maximal between R'' and T_L. Now C has to contain one extra cop for every vertex of R'' not threatened by some T_L. Let T_R be set of these vertices. The cops threatening T_R in M_R can be positioned on the vertices they threaten.

Now we can take $C := V(L) \cup V(R) \cup T_L \cup T_R$.

Case $L' \geq R'$. Let $D := V(R', L') - N[V(L) \cup V(R)]$ (the vertices left to be dominated). Order the vertices of $D \cup L''$ left-to-right by the right ends as $d_1, d_2, \ldots$. We show that we may assume that the cops of C dominating D and threatening R'' are in the positions generated by the following algorithm:

Start with $C' := \emptyset$, $M' := \emptyset$ and take the vertices d_i in turn. If $d_i \in R''$ and not threatened in M', place a cop into C on the right-most neighbour of d_i and add the matching into M'. Otherwise if $d_i \notin N[C']$, add the right-most neighbour (wrt. right endpoint) v of d_i to C'. If there is $r \in R''$ non-threatened neighbour of v, add the left-most such r (wrt. right endpoints) to M' to threaten d_i. In other cases, proceed with next d_{i+1}.

Take the left-most cop c_i of C threatening R'' or dominating D positioned differently than the corresponding c'_i in C'. If c_i is left of c'_i (wrt. right endpoint), we may move c_i right to c'_i. This preserves domination and threatening R'' and (optional) threatening L'' (as c'_i can not be to the right of L'). If c_i is right of c'_i (wrt. right endpoint), then the vertex d_j that caused c'_i to be in C' is either not dominated (if the reason was domination of d_j) or not threatened (if the reason was threatening some $r \in R''$).

Again, we may assume that M is maximal between C' and L''. Then C has to contain extra cops T_L to threaten L'', these cops may be assumed to be on the threatened vertices. Finally, we can take $C := V(L) \cup V(R) \cup C' \cup T_L$. □

7 Conclusion

We have shown an algorithm deciding the Cop and ∞-fast Robber game on interval graphs, therefore answering an open question of Fomin et al. posed in their paper "Pursuing a fast robber on a graph" [4].

Since the game is already NP-hard for general chordal graphs and even split graphs, it would be interesting to consider the complexity of the game on chordal graphs with bounded asteroidal number (or the number of leaves of the underlying tree for the standard intersection representation of chordal graphs) and the class of circular-arc graphs.

The notion of playgrounds of the reduced game can be extended to such graphs and seem to have some common properties, but the analysis does not extend in a straightforward way.

We propose the complexity of the game on such graphs as an open question. Even an algorithm exponential in the asteroidal number would be of interest.

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