Cop-Win Graphs with Maximal Capture-Time

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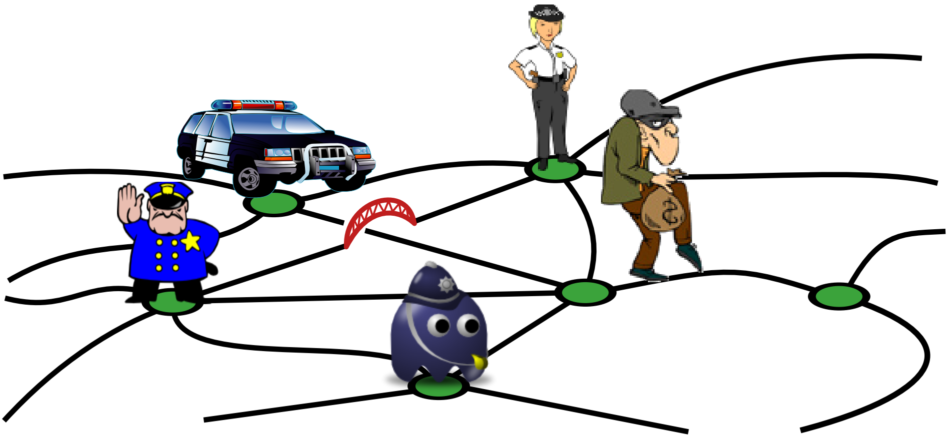
How to catch a robber (on a graph)?



Motivation

Why research cop&robber games on graphs?

- These games are simple and elegant, yet interesting.
- Relations to complexity of some problems on graphs.
- Some games characterize well-known properties in new ways (tree-width, kelly-width, . . .).

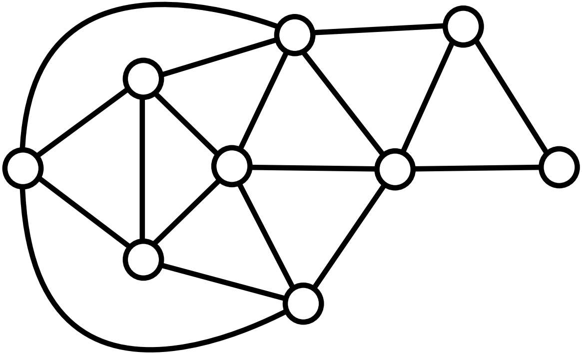
Rules of the Game

Cop&robber is a combinatorial game for two players (the cop and the robber) with complete information played on a given undirected graph G.

- The cop (first) and the robber (second) choose their starting vertices.
- One *turn* is a move to a neighbor vertex or a pass
- Players alternate in turns, starting with the cop
- The cop wins by catching the robber
- The robber wins by avoiding capture indefinitely

A sample game

Starring: the (**C**op), the (**R**obber)



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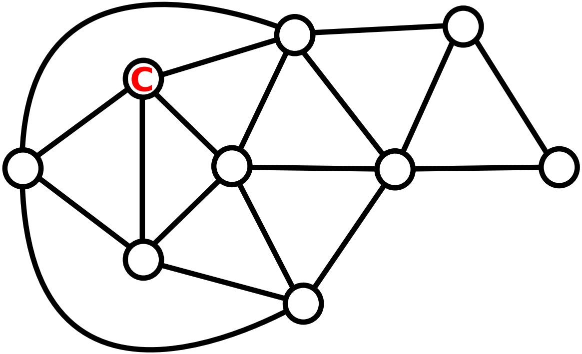

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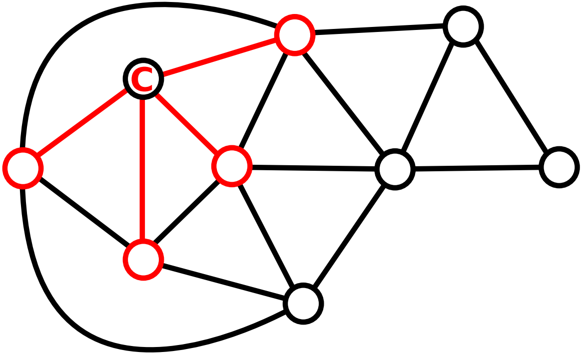

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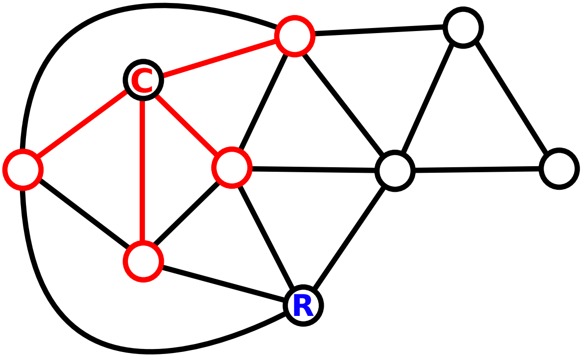

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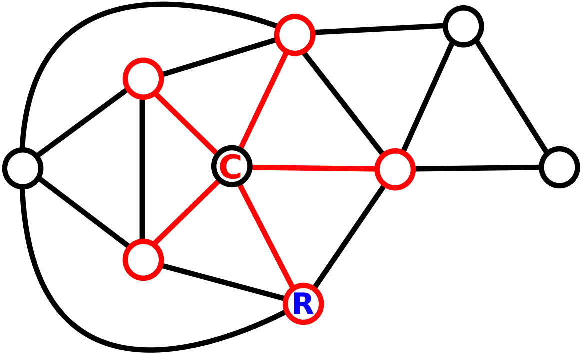

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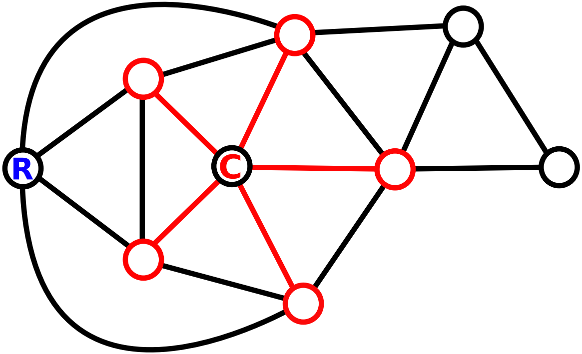

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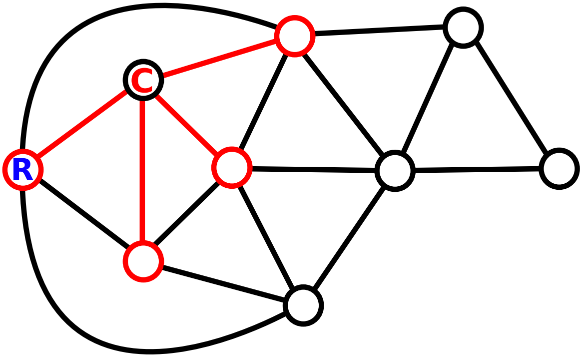

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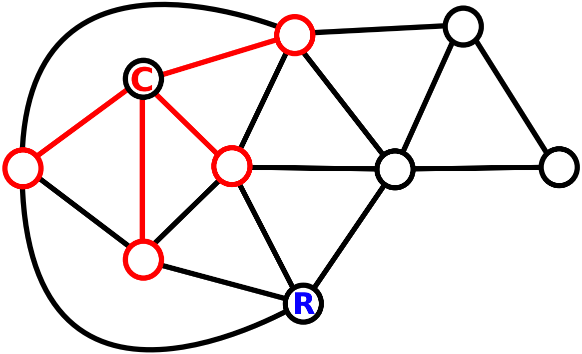

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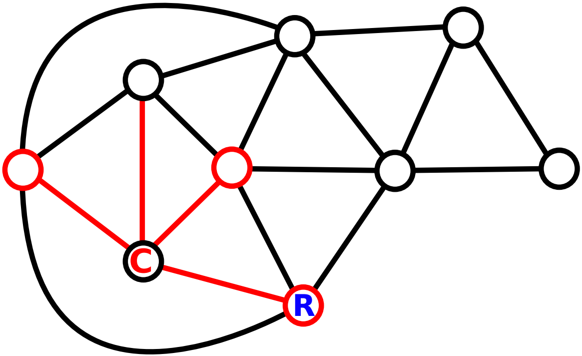

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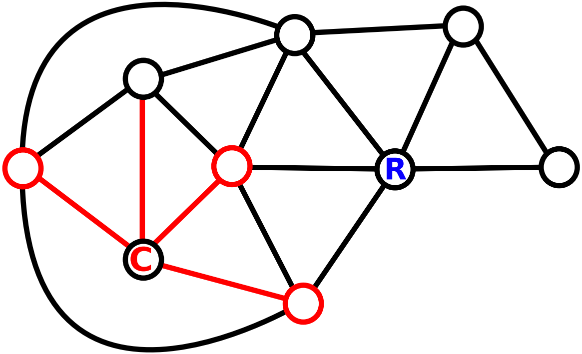

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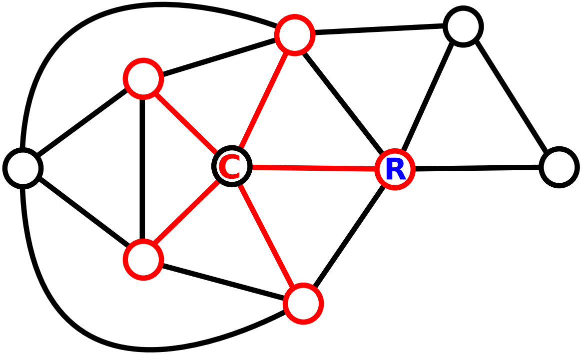

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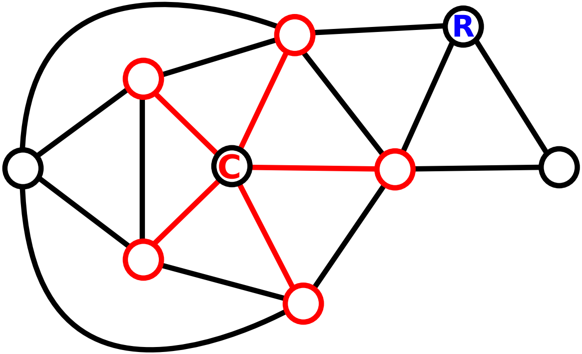

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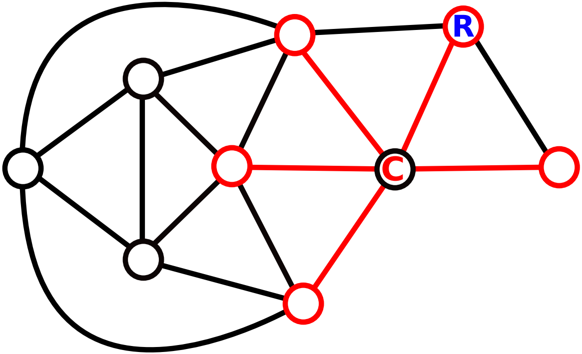

...and the robber is captured.

Facts & Definitions

Fact: On every graph, either the cop or the robber has a winning strategy.

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The game length (wit both players playing optimally) is called the capture-time.

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The function $\mathtt{ct}_{\mathtt{max}}(n)$ is the maximal capture-time of a cop-win graph on n vertices.

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Definition

Let $\mathcal{M}$ be the class of graphs having the maximal capture-time among all the graphs of the same size.

History

- Originally searching cave systems for moving targets.

Nowakowski and Winkler (1983); Quilliot (1978, 1983)

Problem formulation and analysis, the goal is the *number of searchers*.

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- For concrete variants and graph classes also *capture-time*.

Bonato, Golovach, Hahn, Kratochvíl (2006)

Bounds on the capture-time in the paper *The search-time of a graph (to appear in DAM)*.

- Proved $n - 4 \leq \mathtt{ct}_{\mathtt{max}}(n) \leq n - 3$.

Main results

We managed to:

Get the exact upper bound $\mathtt{ct}_{\mathtt{max}}(n)$.

Closed the gap from previous $n - 4 \leq \mathtt{ct}_{\mathtt{max}}(n) \leq n - 3$ to $\mathtt{ct}_{\mathtt{max}}(n) = n - 4$.

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Characterize the structure of the class $\mathcal{M}$.

Effective inductive construction of the entire class.

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Characterize the structure of the class $\mathcal{M}$.

Effective inductive construction of the entire class.

Show that $\mathcal{M}$ is exponentially big.

$\mathcal{M}$ contains at least (2^{n-8}) graphs on n vertices.

The structure of cop-win graphs

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Vertex u dominates a vertex v if every neighbor of v (incl. v) is also a neighbor of u.

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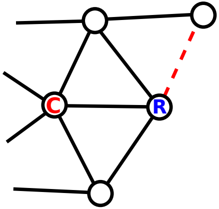
Observation

The robber can not be caught in a non-dominated vertex, the robber can always move out of cop's neighborhood.



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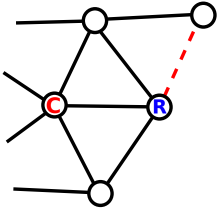

Situation just before capture

Characterization of cop-win graphs [Nowakowski, Winkler]

Graph G is cop-win iff G can be disassembled by consecutive removal of dominated vertices.

The exact value of $\mathtt{ct}_{\mathtt{max}}$

Main Theorem

$\mathtt{ct}_{\mathtt{max}}(n) = n - 4$ for $n \geq 7$, $\mathtt{ct}_{\mathtt{max}}(n) = \lfloor \frac{n}{2} \rfloor$ for $n \leq 7$.

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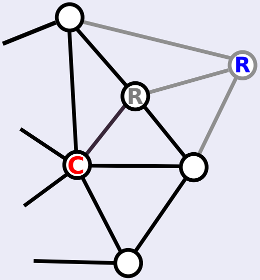
Upper bound on $\mathtt{ct}_{\mathtt{max}}$

- By removing a dominated vertex the capture-time drops by at most 1.



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- It suffices to check that $\mathtt{ct}_{\mathtt{max}}(7) \leq 3$, but there are many cop-win graphs on 7 vertices ...

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- Examine all cop-win graphs from $\mathcal{M}$ on 6 vertices.
  $\mathtt{ct}_{\mathtt{max}}(6) = 3 = n - 3$.
- None can be extended (by appending a dominated vertex) to have a higher capture-time.

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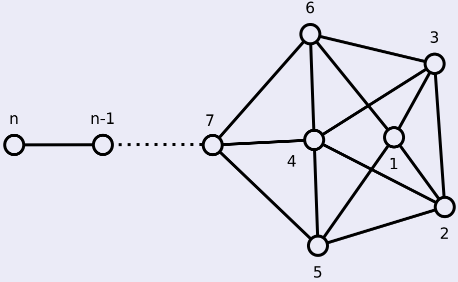
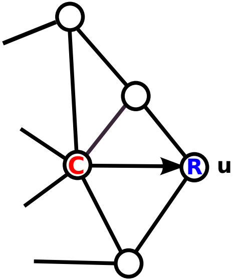
- For $n \geq 7$ we use graphs H_n:



Bonato et. al. had more complicated graphs.

The structure of $\mathcal{M}$



For all the graphs $G \in \mathcal{M}$ (on ≥ 8 vertices) we have:

- G is an extension of $G' \in \mathcal{M}$ by a dominated vertex.

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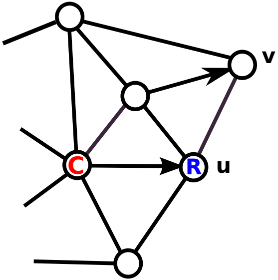

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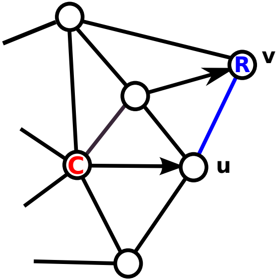

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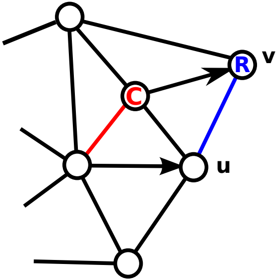

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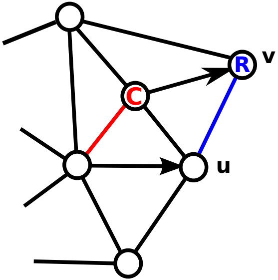

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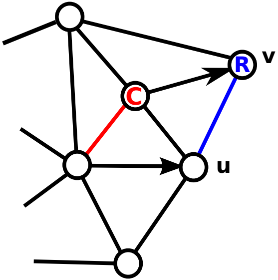

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- Therefore G contains H_n as a subgraph (uniquely determined up to symmetry of H_7).

Cop-win graphs on 8 vertices

- All graphs from $\mathcal{M}$ on ≥ 9 vertices can be constructed by appending dominated vertices.
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- It suffices to determine all the small basic graphs.
- There are too many cop-win graphs on 8 vertices.
- Use of computational power:
  - Program *Nauty* for generating all the non-isomorphic graphs
  - Marking the game state space to determine the capture-time

Result

All the graphs of $\mathcal{M}$ on 8 vertices are extensions of H_7.

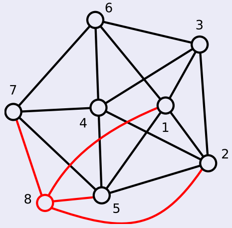
The exponential size of $\mathcal{M}$

Observation

Every graph from $\mathcal{M}$ is constructed uniquely (up to the symmetry of H_7).

Sketch of an algorithm:

- As a base take an asymmetric extension of H_7.



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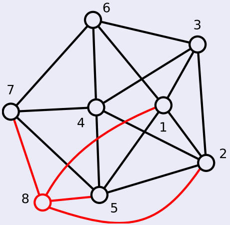

- All the future dominating vertices will be from H_6.
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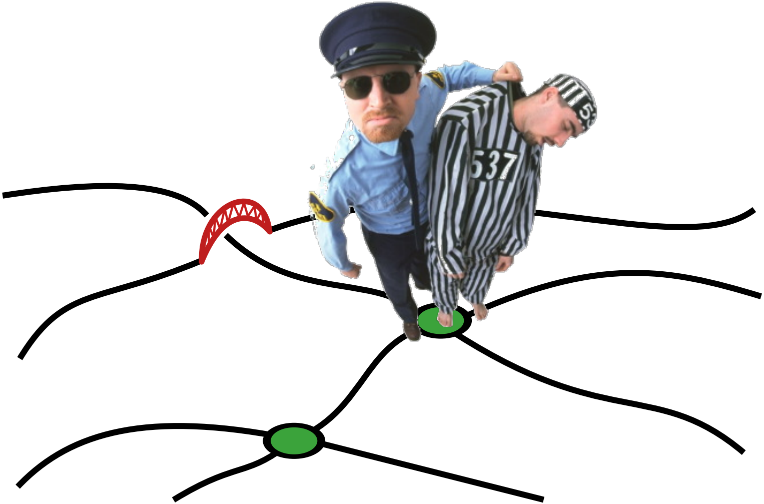
- In $\mathcal{M}$ there are at least 2^{n-8} graphs of size n.
- . . . but the actual number is much higher.

The end?



Thank you for your attention!