Cops and Robbers on Intersection Graphs

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Motivation for pursuit games

Environment

An undirected graph (network, building map, cave system, . . .)

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Participants

- The pursuers, cleaners, rescuers, ... "The cops"
- The fugitive, contamination, ... "The robber"

Goals

- The cops want to capture the robber in finite time
- The robber wants to keep escaping indefinitely

Motivation for pursuit games

Framework of *combinatorial game theory* for formalisation of such games:

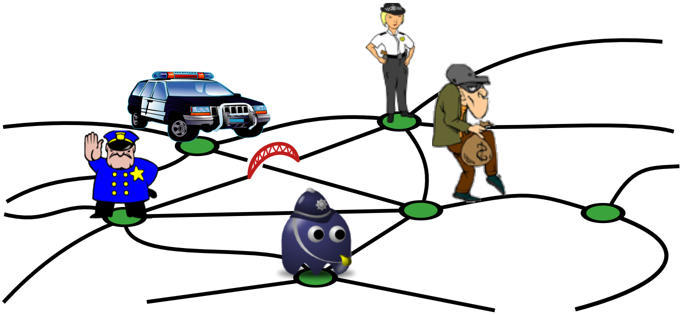
- Two players (no diplomacy)
- Deterministic game and rules (no ambiguity or randomness)
- Complete information (no secrests)
- Discrete turns (no "reaction speed" issues)

Theorem

In this setting, one player always has a deterministic memory-less non-losing strategy.

Cops and robber game

One player controls k cops, the other one one robber; moving on vertices of graph G.



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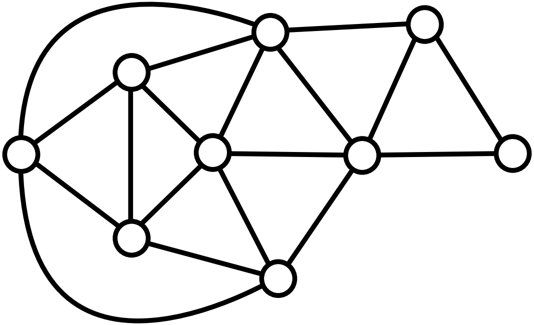
Victory

- The cops win if a cop shares a vertex with the robber.
- The robber wins by escaping indefinitely.

Cops and robber game: example



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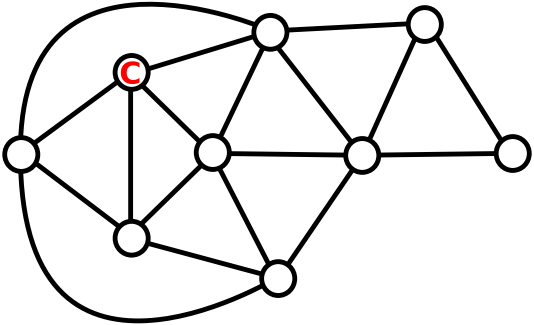

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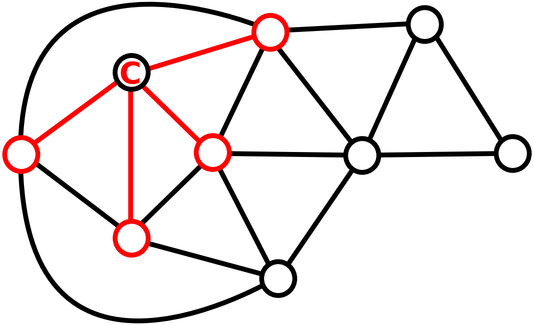

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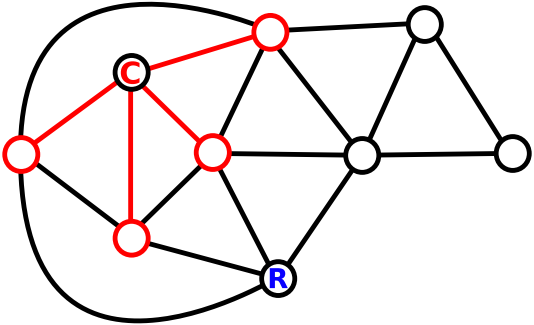

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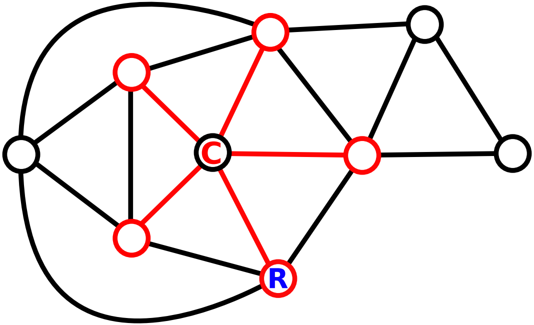

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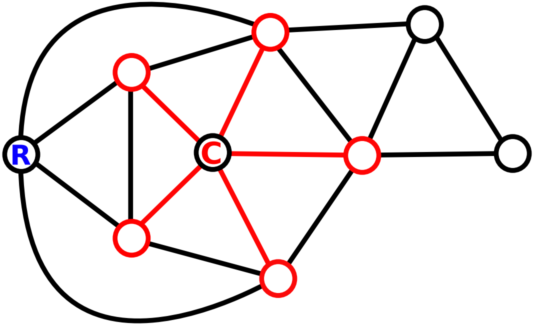

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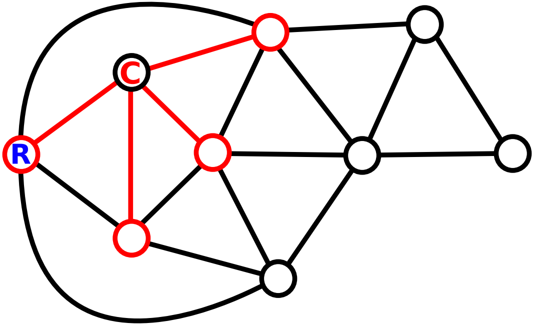

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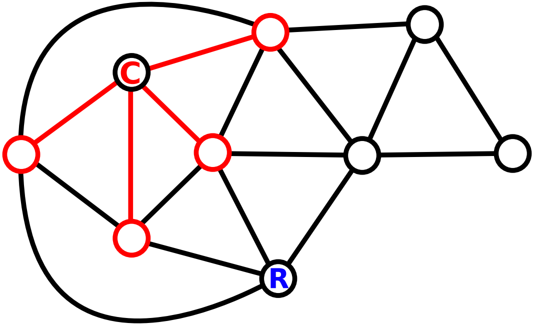

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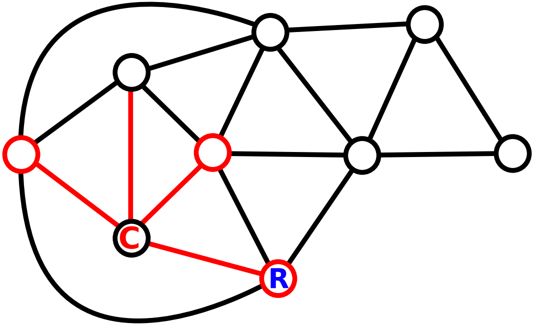

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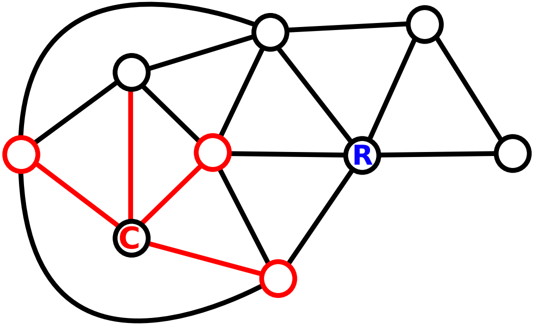

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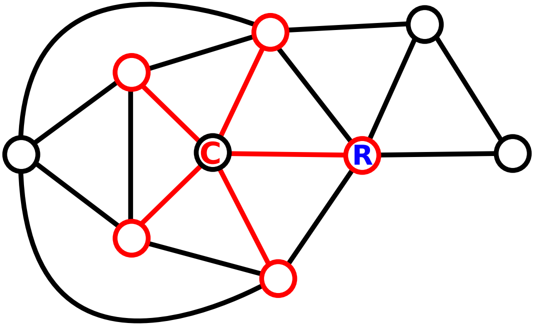

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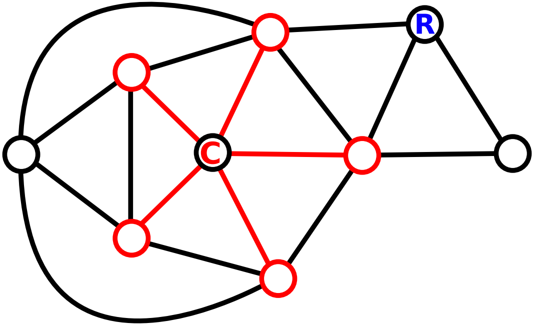

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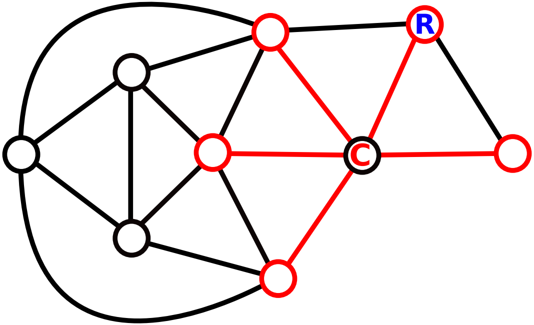

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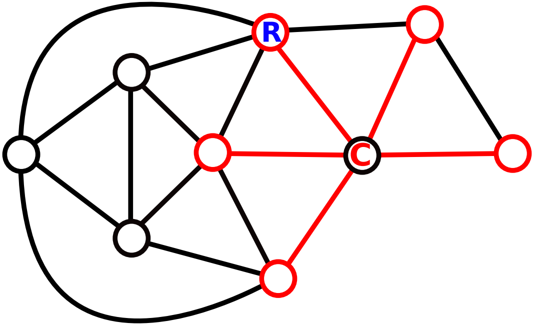

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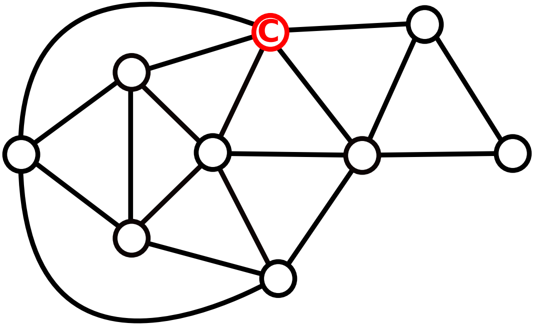

Cops and robbers: background

This game was already studied by Nowakowski, Winkler and Quilliot in 1979 and 1983, motivated by rescuing cave-explorers.

Cop-number $\mathrm{cn}(G)$

The least number of cops having a winning strategy on G.

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Cop-number $\mathrm{cn}(G)$

The least number of cops having a winning strategy on G.

Known results

- Graphs with $\mathrm{cn}(G) = 1$ [N., W. Q. 1983]
- Unbounded $\mathrm{cn}(G)$ in general
- For planar graphs $\mathrm{cn}(G) \leq 3$ [Aigner, Fromme 1984]
- Computing $\mathrm{cn}(G)$ is polynomial for fixed k
- Computing $\mathrm{cn}(G)$ is EXPTIME-complete [Kinnersley 2013]

Cops and robbers: related games

Width parameter characterisation games

- *Tree-width:* Cops with helicopters
- *Path-width:* Invisible robber (occupies a vertex subset)
- *Tree-depth:* Permanent cops, measuring game length
- *Generalised hypertree-width:* Marshalls on hypergraphs

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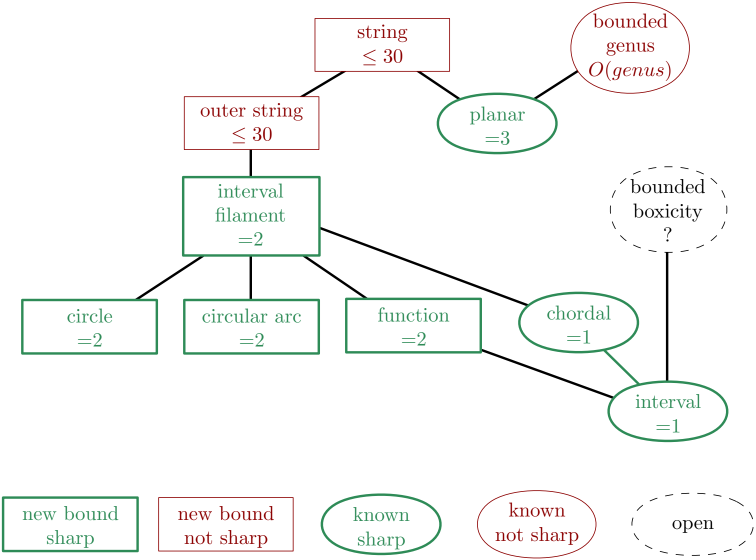
Games with contamination

- Equivalent to invisible robber.
- Cops block spread of contamination.
- Slow vs. fast contamination spread
- Variants with edge ("pipes") or vertex ("cities") contamination.

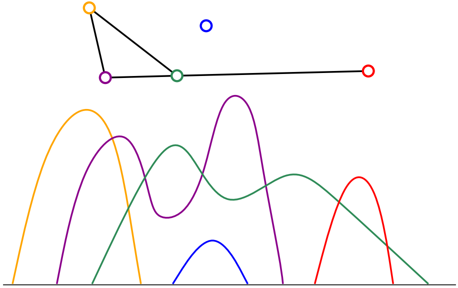
And others . . .

Cop-number bounds on some classes



IFA graphs [Gavril 2000]

Intersection graphs of *interval filaments*.



C&R on IFA graphs

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Theorem (GJKK 2013)

On IFA graphs we have $cn(G) \leq 2$*.*

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Note: This also implies $\mathrm{cn}(G) \leq 2$ for circle graphs, circular arc graphs and function graphs.
In all cases, two cops may be neccessary.

Theorem (Aigner, Fromme 1984)

In any graph, one cop can prevent the robber from entering a given shortest path P.

Idea: The cop stays on the vertex of P "closest" to robber.

C&R on planar graphs

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For planar graphs, $cn(G) \leq 3$. This bound is sharp.

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Idea: Guarding paths to surround robber.

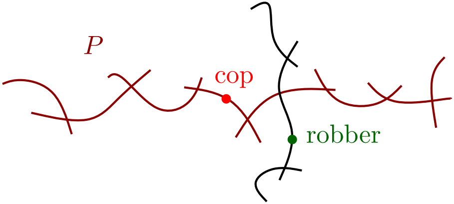
Theorem

For genus-k graphs, $cn(G) \leq 2k + 3$.

Note: This is not known to be sharp.

Guarding paths: string graphs

In string graphs, guarding P is *not* sufficient to prevent robber from crossing P.



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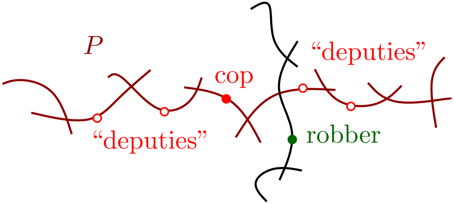

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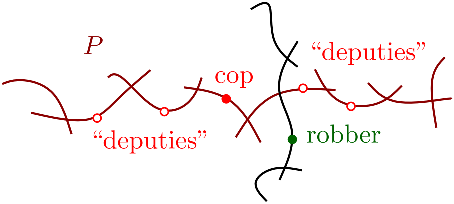

Solution: Also guard two previous and following vertices of P.

Theorem (GJKK 2013)

In any graph, five cops can prevent the robber from entering neighbourhood of a given shortest path P.

Note: Five cops are necessary (when restricted to P).

C&R on string graphs

Theorem (GJKK 2013)

For any string graph G we have $cn(G) \leq 30$.

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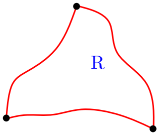

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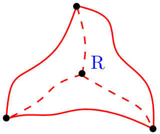

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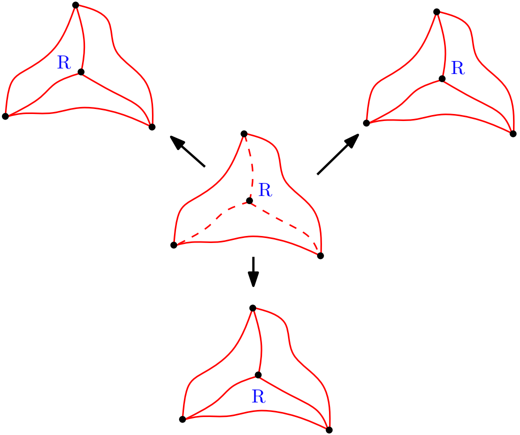

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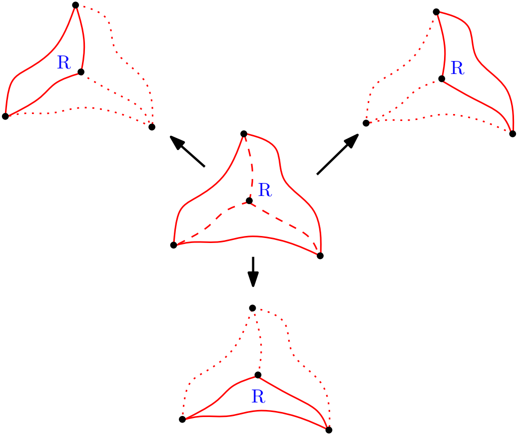

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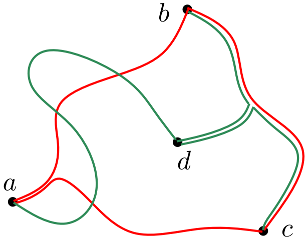
Unfortunately, string graphs may have complicated shortest paths:



Conclusion

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Open problems

- Bound $\mathrm{cn}(G)$ of other graph classes.
- Sharp bounds for $\mathrm{cn}(G)$ of genus-k graphs.
- Better bounds for $\mathrm{cn}(G)$ of string and outer-string graphs.
- Give better algorithms for $\mathrm{cn}(G)$ than observed above.

Thank you!