Cops and Robbers on String Graphs*

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Abstract. The game of cops and robber, introduced by Nowakowski and Winkler in 1983, is played by two players on a graph. One controls k cops and the other a robber. The players alternate and move their pieces to the distance at most one. The cops win if they capture the robber, the robber wins by escaping indefinitely. The cop number of G is the smallest k such that k cops win the game.

We extend the results of Gavenčiak et al. [ISAAC 2013], investigating the maximum cop number of geometric intersection graphs. Our main result shows that the maximum cop number of string graphs is at most 15, improving the previous bound 30. We generalize this approach to string graphs on a surface of genus g to show that the maximum cop number is at most $10g + 15$, which strengthens the result of Quilliot [J. Combin. Theory Ser. B 38, 89–92 (1985)]. For outer string graphs, we show that the maximum cop number is between 3 and 4. Our results also imply polynomial-time algorithms determining the cop number for all these graph classes.

1 Introduction

The Cops and Robber game on graphs has been introduced by Winkler and Nowakowski [12] and independently by Quilliot [14]. In this paper, we investigate this game on the classes of geometric intersection graphs.

Rules of The Game. In this game two players alternate their moves. The first player (called "the cops") places k cops on the vertices of a graph G. Then the second player (called "the robber") chooses a vertex for the robber. Then the players alternate. In the cops' move, every cop either stays in its vertex or moves to one of its neighbors. More cops may occupy the same vertex. In the robber's move, the robber either stays in its vertex, or goes to a neighboring vertex.

The game ends when the robber is *captured* which happens when a cop occupies the same vertex as the robber. The cops wins if he is able to capture the robber. The robber wins if he is able to escape indefinitely.

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Definition 1.1. *For a graph G, its* cop number $\text{cn}(G)$ *is the least number k such that the cops have a winning strategy on G with k cops. For a class of graphs $\mathcal{C}$, the* maximum cop number max-cn$(\mathcal{C})$ *is the maximum cop number* $\text{cn}(G)$ *of a connected graph $G \in \mathcal{C}$, possibly $+\infty$.*

The restriction to connected graphs is standard. The reason is that if G has connected components $C_1, \ldots, C_k$, then $\text{cn}(G) = \sum_{i=1}^{k} \text{cn}(C_i)$.

Known Results. Graphs of the cop number one were characterized already by Quilliot [14]. These are the graphs whose vertices can be linearly ordered $v_1, v_2, \ldots, v_n$ so that each v_i for $i \geq 2$ is a corner of $G[v_1, \ldots, v_i]$, i.e., v_i has a neighbor v_j for some $j < i$ such that v_j is adjacent to all other neighbors of v_i.

For k part of the input, deciding whether the cop number of a graph is at most k has been shown to be NP-hard (2010) [4], PSPACE-hard (2013) [10] and very recently (2015) even EXPTIME-complete [8], confirming a conjecture of Goldstein and Reingold (1995) [7]. In order to test whether k cops suffice to capture the robber on an n-vertex graph, we can search the game graph which has $\mathcal{O}(n^{k+1})$ vertices to find a winning strategy for cops. In particular, if k is a fixed constant, this algorithm runs in polynomial time.

For general graphs on n vertices, it is known that at least $\sqrt{n}$ cops may be needed (e.g., for the incidence graph of a finite projective plane). Meyniel's conjecture states that the cop number of a connected n-vertex graph is $\mathcal{O}(\sqrt{n})$. For more details and results, see the recent book [2].

Geometrically Represented Graphs. Figure 1 shows that the geometry of a graph class heavily influences the maximum cop number. For planar graphs, the classical result of Aigner and Fromme [1] shows that the maximum cop number is 3. This result was generalized to graphs of bounded genus by Quilliot [14] and improved by Schroeder [15]. However, while for planar graphs (genus 0) the maximum cop number is equal 3, already for toroidal graphs (genus 1) the exact value is not known.

We study *intersection representations* in which a graph G is represented by a map $\varphi : V \to 2^X$ for some ground set X such that the edges of G are described by the intersections: $uv \in E \iff \varphi(u) \cap \varphi(v) \neq \emptyset$. The ground set X and the images of φ are usually somehow restricted to get particular classes of intersection graphs. For example, the well-known interval graphs have $X = \mathbb{R}$ and every $\varphi(v)$ a closed interval.

All of these classes admit large cliques, so their genus is unbounded and the above bound of the maximum cop number does not apply. On the other hand existence of large cliques does not imply big maximum cop number since only one cop can guard a maximal clique. It was shown by Gavenčiak et al. [6] that for most of these intersection graph classes, the maximum cop-number is bounded.

In particular, it is shown in [6] that the maximum cop number of string graphs is at most 30. The class of *string graphs* (STRING) is the class of intersection graphs of *strings*: $X = \mathbb{R}^2$ and every $\varphi(v)$ is required to be a finite curve that is a continuous image of the interval $[0, 1]$ in $\mathbb{R}^2$. It is known that every intersection graph of *arc-connected sets* in the plane (i.e., connected regions bounded by closed simple Jordan curves) is a string graph, so the above bound applies to

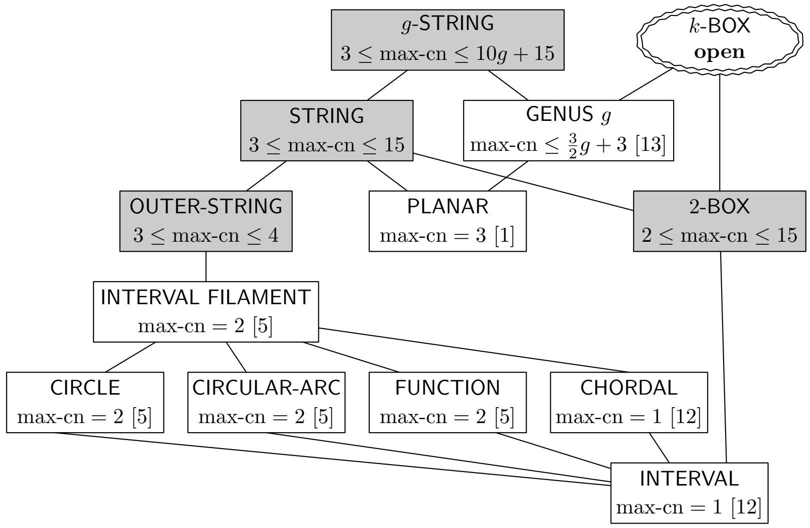

Fig. 1. The Hasse diagram of inclusions between important classes of geometrically represented graphs with known bounds on the maximum cop number. The bounds presented in this paper are in gray.

most classes of intersection graphs in the plane. For instance, *boxicity k graphs* (k-BOX) are intersection graphs of k-dimensional intervals in $\mathbb{R}^k$ and they are string graphs for $k \leq 2$.

Let $\mathbf{S}$ be an arbitrary surface of genus g. We consider a generalization of string graphs for $X = \mathbf{S}$, and we denote this class by g-STRING. It is known that every graph embeddable to a genus g can be represented by a contact representation of disks on a suitable Riemann surface of genus g; so GENUS $g \subsetneq g$-STRING. The class of *outer-string graphs* (OUTER-STRING) consists of all string graphs having string representations with each string in the upper half-plane, intersecting the x-axis in exactly one point, which is an endpoint of this string.

Theorem 1.2. *We show the following bounds for the maximum cop number:*

(i) $3 \leq$ max-cn(OUTER-STRING) ≤ 4.
(ii) $3 \leq$ max-cn(STRING) ≤ 15.
(iii) $3 \leq$ max-cn(g-STRING) $\leq 10g + 15$.

We note that the strategies of cops in all upper bounds are geometric and their description is constructive, using an intersection representation of G. If only the graph G is given, we cannot generally construct these representations efficiently since recognition is NP-complete [9] for string graphs and open for the other classes. Nevertheless, since the state space of the game has $\mathcal{O}(n^{k+1})$ states and the number of cops k is bounded by a constant, we can use the standard exhaustive game space searching algorithm to obtain the following:

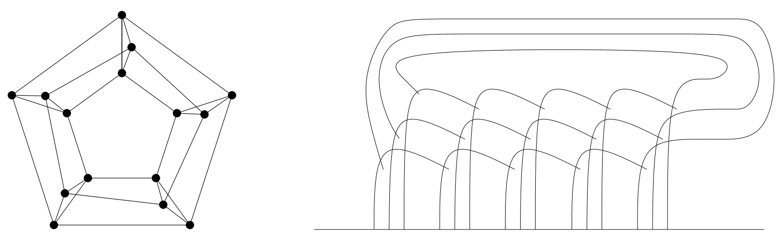

Fig. 2. The 3-by-5 toroidal grid G and one of its outer-string representations. Clearly $\mathrm{cn}(G) = 3$.

Corollary 1.3. *There are polynomial-time algorithms computing the cop number and an optimal strategy for the cops for any outer-string graph, string graph and a string graph on a surface of a fixed genus g.*

Furthermore, our results can be used as a polynomial-time heuristic to prove that a given graph G is not, say, a string graph, by showing that $\mathrm{cn}(G) > 15$. For instance, a graph G of girth 5 and the minimum degree at least 16 is not a string graph since $\mathrm{cn}(G) > 15$: in any position of 15 cops with the robber on v, at least one neighbor of v is non-adjacent to the cops.

Definitions. Let $G = (V, E)$ be a graph. For a vertex v, we use the *open neighborhood* $N(v) = \{u : uv \in E\}$ and the *closed neighborhood* $N[v] = N(v) \cup \{v\}$. Similarly for $V' \subseteq V$, we put $N[V'] = \bigcup_{v \in V'} N[v]$ and $N(V') = N[V'] \setminus V'$. For $V' \subseteq V$, we denote by $G|_{V'}$ the subgraph of G induced by V'. For assumptions for string representations, see the full version.

2 Outer-String Graphs

In this section, we prove that the maximum cop number of outer string graphs is between 3 and 4, thus establishing Theorem 1.2(i).

Proof (Theorem 1.2(i), sketch). Figure 2 shows a connected outer string graph requiring three cops. It remains to show the four cops are always sufficient.

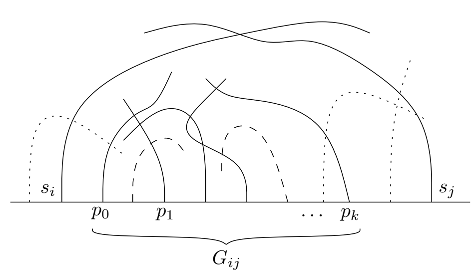

Fig. 3. An overview of the strategy.

We quickly sketch the strategy using Fig. 3, for the details, see the full version. Two cops are called *guards* and in each phase they guard two strings s_i and s_j such that the robber is confined under them. Two other cops called *hunters* travel along strings $p_0, \ldots, p_k$ covering the top of the confined area, always one is on p_i and the other on p_{i+1}. When the robber is confined under p_i and p_{i+1}, the guards move to p_i and p_{i+1} and the next phase begins. □

3 Guarding Shortest Paths and Curves in String Graphs

Shortest Paths. We recall a lemma by Aigner and Fromme [1] giving us a strategy to prevent the robber to enter any given shortest path using only one cop in general graphs.

Lemma 3.1 ([1], Lemma 4). *Let $P = \{u = p_0, p_1, \ldots, p_k = v\}$ be any shortest $u - v$ path. Then a single cop C can, after a finite number of moves (used to move cop C to an appropriate position on P), prevent the robber from safely entering P. That is, if the robber ever moved on P, he would be captured in the next move.*

This result is particularly useful for planar graphs where one can cut the graph by protecting several shortest paths. For intersection graphs, forbidding the robber to visit vertices of P is not sufficient to prevent him from moving from one side of the part to the other. We need a stronger tool to geometrically restrict the robber. We get this by showing that in general graphs we can protect the closed neighborhood of a given shortest path using five cops, preventing the robber from safely stepping on any string even crossing the protected path.

We first need one additional generalization of Lemma 3.1 – we protect paths which are not necessarily shortest in G, but are shortest from the point of the robber within a region he is already confined to. Below we combine this generalization with guarding path neighborhood. We believe that these tools may be of some further interest. We say that an *$u - v$ path P is shortest relative to $D \subseteq V$* if there is no shorter $u - v$ path in G using at least one vertex of D. Note that P itself may or may not go through D.

Confinement, safe moves and guarding. When our strategy makes sure that any time in the future of the game, whenever the robber leaves $D \subseteq V$ he is captured immediately, we say that the robber is *confined to D*. Note that this includes the case when the robber cannot even get outside D without being captured. If the robber can be immediately captured by moving to a vertex v, we say that the robber *cannot safely move to v*. When the strategy makes sure that the robber may not safely move to any $v \in P$, we say that *P is guarded*.

Note that in the following we get exactly the original statement for $D = V$. Also, we could equivalently define a relative shortest path by having no shortcuts contained in D with the same results.

Lemma 3.2. *Let $u, v \in V$ and let P be a shortest $u - v$ path relative to $D \subseteq V$ with the robber confined to D. Then a single cop C can, after a finite number of initial moves, prevent the robber from safely entering P.*

Proof. Since the robber is confined to D, we can apply Lemma 3.1 to $G|_{P\cup D}$. $\square$

Lemma 3.3 ([6], Lemma 6). *Let $u, v \in V$ and let P be a shortest $u-v$ path relative to $D \subseteq V$ with the robber is confined to D. Then five cops $C_{-2}, C_{-1}, C_0, C_1, C_2$ can, after a finite number of initial moves, guard $N[P]$.*

In the following, when we say "start guarding a path", we do not explicitly mention the initial time required to position the five cops onto the path and assume that the strategy waits for enough turns.

Shortest Curves. Since our strategy for string graphs is partially geometric, we introduce the concept of shortest curves as particular curves through the string representation of a shortest path. Note that below we consider any curves sharing only their endpoints to be disjoint.

Let G be a string graph together with a fixed string representation φ, robber confined to $D \subseteq V$ and P a shortest $u-v$ relative to D. Suppose that we choose and fix two points $\pi_u \in \varphi(u)$ and $\pi_v \in \varphi(v)$. Let $\pi_{uv} \subseteq \varphi(P)$ be a curve from π_u to π_v such that $\pi_{uv} \subseteq \bigcup_{p\in P} \varphi(p)$ and for every $p \in P$ π_{uv} has a connected intersection with $\phi(p)$ and these correspond to the points of P in the same order. We call π_{uv} a *shortest curve of P (relative to D)* with endpoints π_u and π_v. A curve π is called a *shortest curve (relative to D)* if it is a shortest curve of some shortest path. We leave out D if $D = V$ or it is clear from the context.

The shortest path in the graph corresponding to a shortest curve π is uniquely defined by the sequence of strings that intersect π on a substring of non-zero length. To *guard a shortest curve* π means to guard its corresponding shortest path. The number of its strings is the *length* of π. Note that the Euclidean length of π plays no role in this paper.

Corollary 3.4. *Let G be a string graph together with a string representation φ and let π be a shortest curve relative to D such that the robber is confined to D. Then five cops can (after a finite number of initial moves) prevent the robber from entering any string intersecting π.*

Proof. Let P be the shortest path such that π is a shortest curve of P. By guarding $N[P]$, the cops prevent from entering strings intersecting π. $\square$

Lemma 3.5. *Any sub-curve (continuous part) of a shortest curve (relative to D) is also a shortest curve (relative to D).* $\square$

4 Capturing Robber in String Graphs

In this section we show that the number of cops sufficient to capture a robber on a connected string graphs is bounded by 15. The idea of the proof of Theorem 1.2(ii) is inspired by the proof of Aigner and Fromme for planar graphs [1]. Before we prove Theorem 1.2(ii), we introduce several notions used below.

Segments, faces and regions. For a given string graph G and its string representation φ let the *faces* (of φ) be the open arc-connected regions of $\mathbb{R}^2 \setminus \varphi(G)$, and let a *closed face* be the closure of a face. As we assume that the number of intersections of φ is finite, the number of faces is also finite. Note that every face is an open set.

A *segment* of string π is a part of the string not containing any intersection with another string between either two intersections, an intersection and an endpoint, or two endpoints. Note that the number of segments is also finite. A *region* is a closed subset of $\mathbb{R}^2$ obtained as a closure of a union of some of the faces.

Let $\mathrm{clos}(X)$ denote the topological closure of a set X and $\mathrm{int}(X)$ the topological interior of X. A vertex v is *internal to B* (also *contained in B*) if $\varphi(v) \subseteq \mathrm{int}(B)$. Denote all vertices internal to a region B by $V_B = \{v \in V \mid \varphi(v) \subseteq \mathrm{int}(B)\}$. In the next section we will use the following topological result, following from the previous section and Corollary 3.4.

Proposition 4.1. *If there is $D \subseteq V$ such that the cops guard disjoint shortest curves π_1 and π_2 (relative to D) between points π_u to π_v such F is the closed face of $\mathbb{R}^2 \setminus (\pi_1 \cup \pi_2)$ containing the robber's string and D is the component of V_F containing the robber, then the robber may not safely leave D.*

Additionally, we use the following topological lemma, for the proof see the full version.

Lemma 4.2. *Given two disjoint simple $\pi_u - \pi_v$ curves π_1 and π_2 in $\mathbb{R}^2$, with π_u, π_v different points, let F be one of the closed faces of $\mathbb{R}^2 \setminus (\pi_1 \cup \pi_2)$. For any simple $\pi_u - \pi_v$ curve π_3 contained in F and going through at least one of its inner points we have that every face of $F \setminus (\pi_1 \cup \pi_2 \cup \pi_3)$ is bounded by simple and disjoint curves π'_i and π'_3 with $\pi'_i \subseteq \pi_i$, $\pi'_3 \subseteq \pi_3$ and $i \in \{1, 2\}$.*

Restricted graphs and strategies. Given a closed region $B \subseteq \mathbb{R}^2$, let G *restricted to B*, denoted $G|_B$, be the intersection graph of the curves of $\varphi \cap B$. This operation may remove vertices (for entire strings outside B), remove edges (crossings outside B) and it also splits each vertex v whose string $\varphi(v)$ leaves and then reenters B at least once. In the last case, every arc-connected part of $\varphi(v) \cap B$ spans a new vertex v_i. The new vertices are also called the *splits* of v. The new graph is again a string graph with representation denoted $\varphi|_B$ directly derived from φ. Note that this operation preserves the faces and strings in $\mathrm{int}(B)$ and all representation properties assumed above, namely the vertex set of $G|_B$ is finite. Also, the number of segments does not increase.

Lemma 4.3. *Let B be a region. If π is a shortest curve (optionally relative to D with the robber confined to D) and $\pi' \subseteq \pi$ is a sub-curve with $\pi' \subseteq B$, then π' is a shortest curve (relative to D) in $G|_B$ and $\varphi|_B$.*

Proof. This follows from Proposition 3.5 and the fact that underlying path of π' is preserved (and if any $p \in P'$ got split into $\{p_i\}$ we use the p_i intersecting π') and no path (e.g. through D) can get shortened by a restriction. ☐

We now show that a strategy for a restricted graph may be used in the original graph.

Lemma 4.4. *Let B be a region. If there is a cop's strategy $\mathcal{S}'$ eventually capturing a robber in $G|_B$ confining him to V_B then there is a strategy $\mathcal{S}$ for the same number of cops capturing the robber on G confining him to V_B.*

Proof. The strategy $\mathcal{S}$ plays out as $\mathcal{S}'$ except when $\mathcal{S}'$ would move a cop to a split $v_i \in V_{G|_B}$ of $v \in V_G$, $\mathcal{S}$ moves the cop to v. Note that all such moves are possible. Robber's choices while internal to B are not extended in any way.

Assume the robber moves from internal u to non-internal v, which is split to $v_1, \dots v_k$ in $G|_B$. Note that at least one of $vv_1, \dots v_k$, say v_i, is adjacent to u in $G|_B$, as $\varphi(u)$ has to intersect $\varphi(v)$ in $\mathrm{int}(B)$. Let $\mathcal{S}$ play as $\mathcal{S}'$ would if the robber moved to v_i, capturing him with this move as assumed in the statement. □

The Strategy for 15 Cops. Our strategy proceeds in phases, monotonously shrinking the safe area of the robber. Slightly informally, in every phase the robber is confined to $D \subseteq V$ by either (A) a single cop guarding a cut-vertex separating D from the rest of the graph or (B) two squads of cops guarding two shortest curves forming a simple closed circle. Then we show that we can decrease either the number of the segments of φ or size of D while not increasing the other and get one of the cases again.

Note that it is important that in case (B) the curves form a simple (not self-intersecting) cycle (as the general case has many technical issues). Also, the reader can assume that the robber is always *inside* the cycle, e.g. using circular inversion of φ if not, but it is not necessary.

Proof (Theorem 1.2(ii)). In any situation let $\widetilde{P}$ be the union of currently guarded paths and vertices. Let D be the component of $V \setminus N[\widetilde{P}]$ containing the robber and let $Q = N[\widetilde{P}] \cap N[D]$. Recall that whenever the robber would be in $N[\widetilde{P}]$ he would be immediately captured, so D is well defined. Let s be the number of segments of φ.

We build a strategy that confines the robber to D for the rest of the game. Therefore we may assume that $V = D \cup Q \cup \widetilde{P}$ as the vertices outside $N[D]$ are irrelevant for the robber and unused by our strategy.

Claim. Let $V = D \cup Q \cup \widetilde{P}$, the robber be on $r \in D$ and one of the following:

(A) One cop guards a vertex c, $|\widetilde{P}| = 1$.

(B) The cops guard two shortest curves π_1 and π_2 (relative to D) between points π_u to π_v such that $\pi_1 \cup \pi_2$ forms a simple cycle, $|\widetilde{P}| \geq 2$ and additionally $G = G|_F$ where F is the closed face of $\mathbb{R}^2 \setminus (\pi_1 \cup \pi_2)$ containing $\varphi(r)$.

Then 15 cops have a strategy to capture the robber confining him to D.

Proof (Claim). We prove this by induction on s and $|D|$, the claim obviously holds for $s \leq 1$ and $|D| = 0$. We distinguish two cases:

Case A. If $Q = \{q\}$ then start guarding q, stop guarding c and let $G' = G - c$ while also leaving out any irrelevant vertices to have $V' = D' \cup Q' \cup \{q\}$ as above. We then use claim case (a) for G' with both smaller s' and $D' \subsetneq D$.

If $Q = \{q_1, \dots q_k\}$, $k \geq 2$, let $G' = G - c$ and let π_{q_i} be any point of $\varphi(c) \cap \varphi(q_i)$. Now let π_1 be a shortest curve between some π_{q_i} and π_{q_j}. We let $\pi_2 \subseteq \varphi(c)$ be the part of $\varphi(c)$ between π_{q_i} and π_{q_j}.

However, $\pi_1 \cup \pi_2$ may not be a simple cycle. Let $\pi_u = \pi_{q_i}$ and let π_v to be the first point of $\pi_1 \cup \pi_2$ along π_2 going from π_u. Note that if there is no other intersection then $\pi_v = \pi_{q_j}$. Now let π_1' and π_2' be the parts of π_1 and π_2 between π_u and π_v, forming a simple cycle.

Let $G'' = G|_F$ where F is the closed face of $\mathbb{R}^2 \setminus (\pi_1 \cup \pi_2)$ containing $\varphi(r)$. Remove any irrelevant vertices from G'' to have $V'' = D'' \cup Q'' \cup \widetilde{P}''$ as above and use claim case (b) for smaller $D'' \subsetneq D$ (as P_1 has a neighbor in D) and not increased $|s''|$. Note that $\widetilde{P}''$ uses at least one vertex other than c.

Case B. If there is no $\pi_u - \pi_v$ path through a vertex of D then, according to Menger's theorem, there must be a cut-vertex $c \in \widetilde{P} \cup Q$ separating D from $\widetilde{P}$. Let one cop guard c and then stop guarding $\widetilde{P}$. Let $G' = G \setminus (\widetilde{P} - c)$ while also leaving out irrelevant vertices to have $V' = D' \cup Q' \cup \{c\}$ as above. We then use claim case (a) for G' with smaller s' and $D' \subseteq D$.

If there is a $\pi_u - \pi_v$ path through a vertex of D, let π_3 be shortest such curve. Note that it is a shortest curve relative to D. Let five cops start guarding π_3 and then let F be the closed face of $\mathbb{R}^2 \setminus (\pi_1 \cup \pi_2 \cup \pi_3)$ containing the robber string. According to Lemma 4.2 we have that F is delimited by disjoint π_i' and π_j' where $i = 3$ or $j = 3$ and $\pi_i' \cup \pi_j'$ form a simple cycle. We let the cops stop guarding π_k where $k \notin \{i, j\}$ and restrict the guarding of π_i and π_j to π_i' and π_j' as in Proposition 3.5. $|\widetilde{P}| \geq 2$ as one vertex string can not form a closed loop.

Let $G' = G|_F$ while also removing any irrelevant vertices from G' to have $V' = D' \cup Q' \cup \widetilde{P}'$ as above. We then use claim case (b) for G' with non-increased s and $D' \subsetneq D$. $\diamond$

Having proven the claim, the theorem then follows by guarding an arbitrary vertex c with one cop so $\widetilde{P} = \{c\}$, defining D and Q as before the claim, and discarding irrelevant vertices to get $V' = D \cup Q \cup \widetilde{P}$. We then use claim case (a) with $G' = G|_{V'}$. $\square$

5 String Graphs on Bounded Genus Surfaces

In this section, we generalise the results of the previous section and prove that $10g + 15$ cops are sufficient to catch the robber on graphs having a string representation on a surface of genus g.

We assume familiarity with basic topological concepts related to curves on surfaces, such as genus, non-contractible closed curves and the fundamental group of surfaces; otherwise see [13]. Specifically, we use the following topological lemma, which directly follows from the properties of the fundamental group.

Lemma 5.1. *Let π_1, π_2 and π_3 be three curves on a surface* **S***, all sharing the same endpoints x and y and oriented from x to y. If the closed curve $\pi_1 - \pi_2$ is non-contractible, then at least one of $\pi_1 - \pi_3$ and $\pi_2 - \pi_3$ is non-contractible as well.* □

Let G be a graph with a string representation φ on a surface **S**. We represent the combinatorial structure of φ by an auxiliary multigraph $A(G)$ embedded on **S** and defined as follows: the vertices of $A(G)$ are the endpoints of the strings of φ and the intersection points of pairs of strings of φ, and the edges of $A(G)$ correspond to segments of strings of φ connecting pairs of vertices appearing consecutively on a string of φ. By representing φ by $A(G)$, we will be able to apply the well-developed theory of graph embeddings on surfaces.

We say that a (closed) walk $W = w_0, w_1, \dots w_k$ in G *imitates* a (closed) curve $\pi \subseteq \varphi[G]$ on the surface **S** if π can be partitioned into a sequence of consecutive segments $\pi_0, \pi_1, \dots, \pi_k$ of positive length, such that $\pi = \sum_{i=0}^{k} \pi_i$ and $\pi_i \subseteq \varphi(w_i)$ for each $i = 0, \dots, k$. A closed walk W *imitates a non-contractible curve* if there is a non-contractible curve $\pi \subseteq \varphi[G]$ imitated by W.

Lemma 5.2. *Let φ be a string representation of a connected graph G on a surface* **S** *of genus $g > 0$ and let W be a closed walk in G imitating a non-contractible curve. Then every connected component of the graph $G' = G - N[W]$ has a string representation on a surface of genus at most $g - 1$.*

Lemma 5.3. *If a graph G has no string representation in the plane, then for every string representation φ of G on a surface* **S** *there is a closed walk W in G imitating a non-contractible curve.*

Proof. Let $A(G)$ be the auxiliary multigraph corresponding to the string representation φ. Since $A(G)$ is not planar, the embedding of $A(G)$ contains a noncontractible cycle (see [11, Chapter 4.2]), which corresponds to a noncontractible curve on **S**. This curve is imitated by a closed walk W of G. □

Lemma 5.4. *On a graph G with a string representation φ on a surface* **S** *and a shortest closed walk W imitating a non-contractible curve, 10 cops have a strategy to guard $N[W]$ after a finite number of initial moves – that is capture a robber immediately after he enters a vertex of $N[W]$.*

Proof (Theorem 1.2(iii)). Let G be a connected graph with a string representation φ on a surface **S** of the smallest possible genus g. We want to show that $10g + 15$ cops have a strategy to capture the robber on G. We proceed by induction on the genus g. If $g = 0$, we use Theorem 1.2(ii).

Let $g > 0$, and fix a string representation of G on a surface of genus g. Let W be a shortest closed walk in G imitating a non-contractible curve.

By Lemma 5.4, 10 cops may, after a finite amount of moves, prevent the robber from entering $N[W]$. The first part of the cops' strategy is to designate a group of 10 cops that will spend the entire game guarding $N[W]$. Thus, after a finite number of moves the robber will remain confined to a single connected component K of the graph $G' = G - N[W]$.

By Lemma 5.2, the graph K has a string representation on a surface of genus at most $g-1$, and by induction, $10(g-1)+15$ cops have a strategy to capture the robber on K. Thus, $10g+15$ cops will capture the robber on G. □

6 Conclusions

In this paper, we improve the bound on the maximum cop number of string graphs and also generalize this bound for string graphs on arbitrary surfaces. It remains open whether other intersection classes of special and higher dimensional sets have bounded maximum cop number. In particular:

Problem 6.1. Is the maximum cop number of k-BOX bounded?

We note that bounded genus graphs have bounded boxicity [3]. If the answer is positive, it implies another strengthening of [14,15].

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