FIREFIGHTING ON SQUARE AND HEXAGONAL GRIDS

TOMÁŠ GAVENČIAK, JAN KRATOCHVÍL, AND PAWEŁ PRAŁAT

Abstract. In this paper, we consider the *firefighter problem* on a graph $G = (V, E)$ that is either finite or infinite. Suppose that a fire breaks out at a given vertex $v \in V$. In each subsequent time unit, a firefighter protects one vertex which is not yet on fire, and then the fire spreads to all unprotected neighbors of the vertices on fire. The objective of the firefighter is to save as many vertices as possible (if G is finite) or to stop the fire from spreading (for an infinite case).

The surviving rate $\rho(G)$ of a finite graph G is defined as the expected percentage of vertices that can be saved when a fire breaks out at a vertex of G that is selected uniformly random. For a finite square grid $P_n \square P_n$, we show that $5/8 + o(1) \leq \rho(P_n \square P_n) \leq 67243/105300 + o(1)$ (leaving the gap smaller than 0.014) and conjecture that the surviving rate is asymptotic to $5/8$.

We define the surviving rate for infinite graphs and prove it to be $1/4$ for the infinite square grid, even in the case of finitely many initial fires. For the infinite hexagonal grid we provide a winning strategy if two additional vertices can be protected at any point of the process, and we conjecture that the firefighter has no strategy to stop the fire without additional help. We also show how the speed of the spreading fire can be reduced by a constant factor.

`Draft version May 30, 2013`

1. Introduction

The following *firefighter problem* on a graph $G = (V, E)$ was introduced by Hartnell at a conference in 1995 [9]. Suppose that a fire breaks out at a given vertex $v \in V$. In each subsequent time unit (called a turn), a firefighter *protects* one vertex which is not yet on fire and then the fire spreads to all unprotected neighbors of the vertices already on fire. Once a vertex is on fire or is protected it stays in such state forever. Protecting a vertex is in essence equivalent to deleting it from the graph.

The game stops if no neighbor of the vertices on fire is unprotected and the fire cannot spread. If the graph is finite, the game finishes at some point and the goal of the firefighter is to save as many vertices as possible. In case of an infinite graph, the goal of the firefighter is to stop the fire from spreading or, if this is not possible, to save as many vertices as possible in the limit (we define this graph parameter in Section 3).

Today, almost 20 years later, our knowledge about this problem is much greater and a number of papers have been published. We would like to refer the reader to the survey of Finbow and MacGillivray for more information [6].

1991 *Mathematics Subject Classification.* 05C57 .

Key words and phrases. Firefighter, surviving rate, square grid, hexagonal grid.

The first author is supported by GAUK grant SVV-2013-267313.

The second author is supported by Czech Research grant CE-ITI GAČR P202/12/6061.

The third author gratefully acknowledges support from NSERC and Ryerson University.

For finite graphs, we focus on the following property. Let $\mathrm{sn}(G, v)$ denote the number of vertices in G the firefighter can save when the fire breaks out at a vertex $v \in V$, assuming the best strategy is used. Then let $\rho(G, v) = \mathrm{sn}(G, v)/n$ be the proportion of vertices saved (here and throughout the paper, n denotes the number of vertices of G, assuming G is finite). The *surviving rate* $\rho(G)$ of G, introduced in [4], is defined as the expected $\rho(G, v)$ when the fire breaks out at a random vertex v of G (uniform distribution is used), that is,

$$\rho(G) = \frac{1}{n} \sum_{v \in V} \rho(G, v) = \frac{1}{n^2} \sum_{v \in V} \mathrm{sn}(G, v).$$

For example, it is not difficult to see that for cliques $\rho(K_n) = \frac{1}{n}$, since no matter where the fire breaks out only one vertex can be saved. For paths we get that

$$\rho(P_n) = \frac{1}{n^2} \sum_{v \in V} \mathrm{sn}(G, v) = \frac{1}{n^2} \left(2(n-1) + (n-2)(n-2)\right) = 1 - \frac{2}{n} + \frac{2}{n^2}$$

(one can save all but one vertex when the fire breaks out at one of the leaves; otherwise two vertices are burned).

It is not surprising that almost all vertices on a path can be saved, and in fact, all trees have this property. Cai, Cheng, Verbin, and Zhou [1] proved that the greedy strategy of Hartnell and Li [10] for trees saves at least $1-\Theta(\log n/n)$ percentage of vertices on average for an n-vertex tree. Moreover, they managed to prove that for every outer-planar graph G, $\rho(G) \geq 1 - \Theta(\log n/n)$. Both results are asymptotically tight and improved upon earlier results of Cai and Wang [2]. (Note that there is no hope for a similar result for planar graphs, since, for example, $\rho(K_{2,n}) = 2/(n+2) = o(1)$.) However, this does not mean that it is easy to find the exact value of $\rho(G)$. It is known that the decision version of the firefighter problem is NP-complete even for trees of maximum degree three [5].

Moving to another interesting direction, the third author of this paper showed that any graph G with average degree strictly smaller than $30/11$ has the surviving rate bounded away from zero [11] and showed that this result is sharp (the construction uses a mixture of deterministic and random graphs). (See [12] for a generalization of this result for the k-many firefighter problem.) These results improved earlier observations of Finbow, Wang, and Wang [7].

1.1. **Our contribution.** First, we study the surviving rate of $P_n \square P_n$, the Cartesian product of two paths of length $n-1$. It was announced by Cai and Wang that

$$0.625 + o(1) = \frac{5}{8} + o(1) \leq \rho(P_n \square P_n) \leq \frac{37}{48} + o(1) \approx 0.7708$$

but a formal proof has not been published. We will prove the following result, which provides much better upper bound.

Theorem 1. *For the Cartesian product of two paths we have*

$$0.625 + o(1) = \frac{5}{8} + o(1) \leq \rho(P_n \square P_n) \leq \frac{67243}{105300} + o(1) < 0.6386.$$

Our proof for the upper bound is not very sophisticated and there are ways to improve it. One the other hand, it narrows down the surviving rate to a small interval smaller than 0.014. It is natural to conjecture the following but this still remains open.

Conjecture 2. $\lim_{n\to\infty} \rho(P_n \Box P_n) = 5/8$.

For an infinite graph $G = (V, E)$, the primary goal is to determine if the fire can be stopped from spreading or not. All graphs we discuss here are vertex transitive so the choice of the starting point is irrelevant.

It is known (and easy to show) that it is impossible to surround the fire with one firefighter in the infinite Cartesian grid (see [13, 8]). On the other hand, it is clear that two firefighters can stop the fire (that is when two vertices can be protected in each round) and in [13] the optimal strategy was provided that does it in 8 steps. (See [3] for a fractional version of this problem.) It was proved in [8] that if the fire breaks out on the triangular grid, two firefighters do not suffice to surround the fire but three of them do the job easily.

For the infinite square grid $G_\Box$, we show that it is optimal to save a $90°$ wedge of vertices. In Section 3 we formally introduce a measure of the surviving rate for infinite graphs and show that $\rho(G_\Box) = 1/4$.

For the infinite hexagonal grid G_{hex}, the problem is far from being trivial. We show that we can save 2/3 of the grid:

Theorem 3. $\rho(G_{hex}) \geq 2/3$.

With just a little additional help of two extra protected vertices, it is possible to stop the fire from spreading:

Theorem 4. *In case the firefighter is allowed to protect one extra vertex at time t_1 and one at time t_2, $1 \leq t_1 \leq t_2$ (possibly protecting two extra vertices at once if $t_1 = t_2$), there is a strategy that will contain the fire on G_{hex}. The strategy does not depend on the actual value of t_1, respectively t_2, before turn t_1, respectively t_2.*

The proof also shows that one extra protected vertex can help to save at least 5/6 of the hexagonal grid.

We also show a strategy to slow down the fire by a constant factor in the following sense. Here and throughout the paper $N_t(v) = \{x \in V(G) : d_G(x, v) = t\}$ denotes the t-th neighborhood of v in the graph G, and $N_{\leq t}(v) = \{x \in V(G) : d_G(x, v) \leq t\} = \bigcup_{s\leq t} N_s(v)$.

Theorem 5. *There exists $c < 1$ such that when only finitely many vertices of the infinite hexagonal grid are burning, there is $n_0 \geq 0$, $v_0 \in V$ and a strategy such that for every $T \geq n_0$ all vertices burning after turn T are contained in $N_{\leq cT}(v_0)$.*

Finally, even though one can contain the fire with just 2 extra protected vertices and can slow down the fire, we conjecture that one firefighter per turn alone still cannot stop the fire from spreading.

Conjecture 6. *If a fire breaks out on the hexagonal grid, one firefighter does not suffice to contain the fire.*

2. Finite square grid

A *square grid graph* $P_n \Box P_n = (V, E)$ is the graph whose vertices correspond to the points in the plane with integer coordinates from

$$C = \{-\lfloor n/2 \rfloor, -\lfloor n/2 \rfloor + 1, \ldots, -1, 0, 1, \ldots, \lceil n/2 \rceil - 1\}$$

and two vertices are connected by an edge whenever the corresponding points are at distance 1. In other words,

$$\begin{aligned} V &= \{(a,b) : a, b \in C\}, \\ E &= \{vu : v, u \in V \text{ and } \|v - u\| = 1\}. \end{aligned}$$

We prove the lower bound and the upper bound stated in Theorem 1 in two separate subsections.

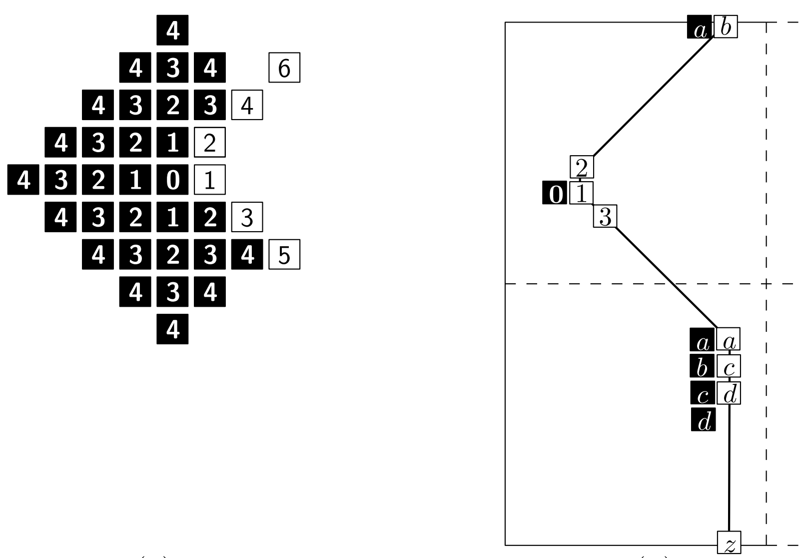
2.1. **Lower bound.** Consider the square grid $P_n \Box P_n$ for some integer n. Suppose that a fire breaks out at a vertex (a, b). Due to the symmetry, we may assume that $0 \le a \le \lceil n/2 \rceil - 1$ and that $0 \le b \le a$. The firefighter can protect the following sequence of vertices in the first few rounds (see Figure 1 (a)):

$$(a-1, b), (a-1, b+1), (a-2, b-1), (a-2, b+2), \ldots$$

Once the *north* border ($b = \lceil n/2 \rceil - 1$) is reached (the last vertex protected is $(a - \lfloor n/2 \rfloor + 1 + b, \lfloor n/2 \rfloor - 1)$), the firefighter goes straight down to the *south* border ($b = -\lfloor n/2 \rfloor$) protecting the sequence

$$(a - \lfloor n/2 \rfloor + b, 2b - \lfloor n/2 \rfloor + 1), (a - \lfloor n/2 \rfloor + b, 2b - \lfloor n/2 \rfloor), \ldots.$$

(For a 'big picture' of this strategy see Figure 1 (b).)



(a) (b)

Figure 1. The beginning of the defense strategy and the 'big picture'. The numbers indicate the turn in which the vertex was lit or protected.

If $a = xn + o(1)$ and $b = yn + o(1)$ for some $0 \le x \le 1/2$ and $0 \le y \le x$. Using the described strategy, it is easy to see what percentage of vertices can be saved, which gives us the following bound.

$$\frac{\mathrm{sn}(P_n \Box P_n, (a,b))}{n^2} \ge \left(\frac{1}{2} - y\right)^2 + \left(\frac{1}{2} + x\right) - \left(\frac{1}{2} - y\right) + o(1) = \left(\frac{1}{2} - y\right)^2 + x + y + o(1).$$

Hence, since there are 8 symmetric regions to consider,

$$\begin{aligned} \rho(P_n \Box P_n) &= \frac{1}{n^2} \sum_{(a,b)\in V} \frac{\mathrm{sn}(P_n \Box P_n, (a,b))}{n^2} \\ &\ge 8 \int_0^{1/2} \int_0^x \left(\left(\frac{1}{2} - y\right)^2 + x + y \right) dy dx + o(1) \\ &= 8 \int_0^{1/2} \left(-\frac{1}{3} \left(\frac{1}{2} - x\right)^3 + \frac{3}{2} x^2 + \frac{1}{24} \right) dx + o(1) \\ &= \frac{5}{8} + o(1). \end{aligned}$$

The lower bound holds.

2.2. **Upper bound.** Consider the square grid $P_n \Box P_n$ for some integer n. For a given vertex (a,b) and $r \in \mathbb{N}$, we abuse slightly the notation and use $N_r(a,b)$ instead of $N_r((a,b))$, the set of vertices at distance r from (a,b). We start with the following simple but very powerful observation.

Lemma 7. *Suppose that a fire breaks out at a vertex (a,b) of $P_n \Box P_n$. Regardless of the strategy used by the firefighter, there are at most r vertices in $N_r(a,b)$ that are not burning at time r, for every $r \ge 1$.*

Proof. In order to prove the theorem, we prove the following stronger claim: *At time $r \ge 1$, for every non-burning vertex v of $N_r(a,b)$ there is a path P_v from v to some protected vertex $p(v)$. It is allowed that a path is trivial (that is, $v = p(v)$) when v is itself protected. Moreover, all the paths P_v are vertex disjoint.*

We prove the claim by induction. Clearly, the property holds for $r = 1$: if a neighbor of (a,b) is not burning at time $r = 1$, then it must be protected. Suppose that the property holds for $r \ge 1$, our goal is to show that it holds for $r+1$. Let v be a vertex of $N_{r+1}(a,b)$ that is not burning at time $r+1$. If v is protected, then it yields a trivial path. Suppose then that v is not protected. It is clear that no neighbor of v was burning in time r; otherwise, v would be on fire in time $r+1$ too. Hence, each neighbor of v in $N_r(a,b)$ is associated with a unique path; v has two such neighbors, unless v is a 'corner' vertex in which case there is only one such neighbor. It follows that paths can be extended to all non-burning vertices of $N_{r+1}(a,b)$ by choosing one extension direction for every *group* of non-burning vertices (see Figure 2). This is always possible *unless* a whole fire-front (one of NE, NW, SE, or SW edges of the diamond $N_r(a,b)$) is a non-burning group. Fortunately, this situation cannot occur since this would imply that there were $r+1$ vertices of $N_r(a,b)$ not burning at time r and therefore $r+1$ paths to $r+1$ protected vertices, which is impossible. (Note that $|N_r(a,b)| = 4r$ but each fire-front consists of

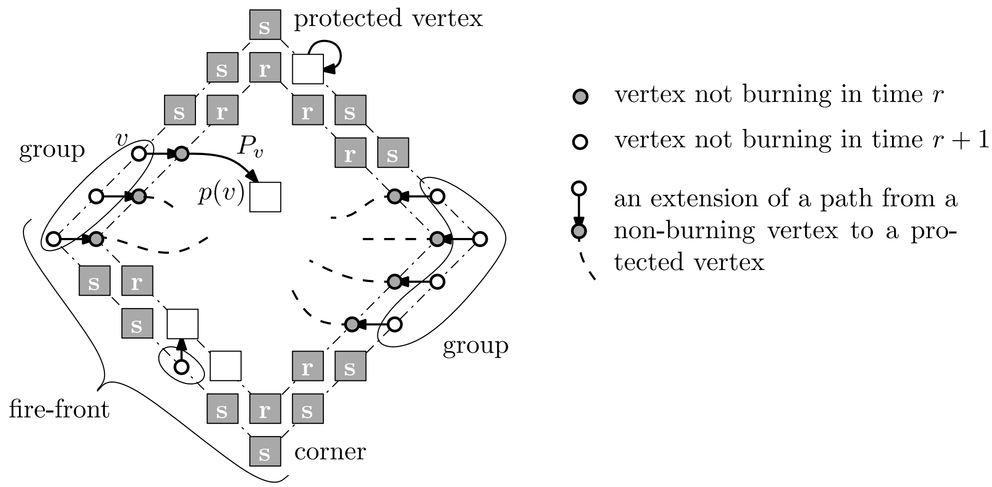


FIGURE 2. Extending paths in time r to time $s = r+1$.

$r+1$ vertices, including two 'corners'.) Therefore the claim holds for $r+1$ and the proof is finished. □

Before we move to investigating the surviving rate of $P_n \square P_n$ let us focus on the case $(a,b) = (0,0)$ in order to explain the idea in a simple setting. Consider first the graph G induced by the set of vertices at distance at most $\lceil n/2 \rceil - 1$ from $(0,0)$ (that is, 'diamond shape' square grid). It follows from Lemma 7 that for every $1 \le r \le \lceil n/2 \rceil - 1$, the fraction of vertices from $N_r(0,0)$ that are saved is at most $1/4$ (since $|N_r(0,0)| = 4r$ for $1 \le r \le \lceil n/2 \rceil - 1$). On the other hand, the strategy for the firefighter provided in the previous subsection guarantees that this can be achieved. Hence, $\mathrm{sn}(G,(0,0))/|V(G)| = 1/4 + o(1)$.

For the original grid $P_n \square P_n$ the situation is slightly more complicated, even for the starting point $(a,b) = (0,0)$. We need to investigate the number of vertices at distance r from $(0,0)$ which changes once we reach the boarder. We have

$$|N_r(0,0)| = \begin{cases} 4r & \text{if } 1 \le r \le \lceil n/2 \rceil - 1 \\ 4(n-r) + O(1) & \text{if } \lceil n/2 \rceil - 1 < r \le 2\lfloor n/2 \rfloor. \end{cases}$$

It follows from Lemma 7 that the number of vertices burnt is at least

$$\begin{aligned} \sum_{r=1}^{2\lfloor n/2 \rfloor} \max(|N_r(0,0)| - r, 0) &\ge n^2 \left(\int_0^{1/2} 3x dx + \int_{1/2}^{4/5} (4-5x) dx + o(1) \right) \\ &= n^2 \left(\frac{3}{5} + o(1) \right). \end{aligned}$$

We get that the fraction of vertices saved is at most $2/5 + o(1)$. Clearly, this bound can be improved. In order to play optimally and save r vertices at distance r during the first phase ($r \le n/2$) the firefighter has to follow the strategy described in the previous subsection. But if this is the case, the strategy is not optimal in the second phase ($r > n/2$) and there is no way to keep saving r vertices at distance r. As we already mentioned, we conjecture that the strategy yielding the lower bound is optimal, giving the following conjecture for the case $(a, b) = (0, 0)$.

Conjecture 8.

$$\lim_{n\to\infty} \frac{\operatorname{sn}(P_n \square P_n, (0,0))}{n^2} = \frac{1}{4}.$$

The proof for the general case is rather technical and we present it with all details in Appendix A.

3. Infinite graphs

In this section we introduce the concept of surviving rate for infinite graphs with all finite degrees, and present a few results for infinite square and hexagonal grid. Assuming a fixed and deterministic firefighter strategy, a vertex v is considered *saved* if the strategy guarantees that v never catches fire (v does not have to get protected during the process). This is well defined even for infinite graphs—given a fixed and deterministic strategy, the game is pre-determined and either there is a point of the process when v catches fire or it is never on fire (that is, v is saved). The surviving rate of a strategy $\mathcal{F}$ (used in the process in which the fire breaks out at vertex v) is then defined as $\rho_{\mathcal{F}}(G, v) = \liminf_{i\to\infty} \frac{|N_{\le i}(v)\cap S|}{|N_{\le i}(v)|}$ where $S = S(\mathcal{F})$ is the set of saved vertices and $N_{\le i}(v)$ is the set of vertices at distance at most i from v. As we assume that all degrees are finite, $|N_{\le i}(v)|$ is always finite. Let the optimal surviving rate be $\rho_{\mathcal{F}}(G, v) = \sup_{\mathcal{F}} \rho_{\mathcal{F}}(G, v)$. We always distinguish whether this ratio can be attained or not. Note that this coincides with the definition of $\rho(G, v)$ for finite graphs.

For example, for an infinite binary tree T_2 rooted at r, we have $\rho(T_2, r) = 1$ as we can save all the vertices but a single infinite ray (path) from r. This follows from the fact that for trees, it is always optimal to protect a neighbor of a burning vertex rather than a vertex further away. Note that it is not possible to save all but finite number of vertices of T_2. Similarly, for an infinite ternary tree T_3 rooted at r we have $\rho(T_3, r) = 1/2$. We omit the proof of this statement and leave it as an exercise.

The expected surviving rate is not extensible to infinite graphs without explicitly stating the distribution (there is no uniform distribution on an infinite number of vertices). However, for vertex-transitive graphs, we have $\rho(G, v_1) = \rho(G, v_2)$ for any two vertices v_1, v_2, and we denote it as $\rho(G)$. Note that both square grid and hexagonal grid are vertex-transitive.

To see the robustness of our definition, let us generalize the definition by allowing a different center of measurement: $\rho_{\mathcal{F}}(G, v, c) = \liminf_{i\to\infty} \frac{|N_{\le i}(c)\cap S|}{|N_{\le i}(c)|}$ where S are the vertices saved by $\mathcal{F}$, provided that the first breaks out at vertex v. Generally, the ratio depends on the choice of c (as can be seen in T_3 and other fast-expanding graphs; in T_3 we can place c to the root), but when $|N_i(c)|$ grows asymptotically strictly slower than $|N_{\le i}(c)|$, we get the following result:

Lemma 9. *Given an infinite connected graph G with finite degrees, let $C_c(i) = |N_i(c)|$ and $A_c(i) = |N_{\le i}(c)|$. If we have $C_c(i) = o(A_c(i))$ (the standard notation $f(i) = o(g(i))$ denotes the fact that $f(i)/g(i) \to 0$ as $i \to \infty$) for some $c \in V(G)$, we have $\rho_{\mathcal{F}}(G, v, c) = \rho_{\mathcal{F}}(G, v, c')$ for any $c' \in V(G)$.*

Note that this is the case for square, hexagonal, triangular and many other grids.

Proof. Assume fixed c, c', $\mathcal{F}$ and S and let $d = d(c', c)$. Then for any i we have

$$\begin{aligned} |N_{\le i}(c') \cap S| &\ge |N_{\le i-d}(c) \cap S| \\ &= |N_{\le i}(c) \cap S| - \sum_{j=i-d+1}^{i} |N_j(c) \cap S| \\ &= |N_{\le i}(c) \cap S| - o(|N_{\le i}(c)|), \end{aligned}$$

since $|N_{\le i}(c)|$ is non-decreasing. Similarly, we get $|N_{\le i}(c') \cap S| \le |N_{\le i}(c) \cap S| + o(|N_{\le i+d}(c)|)$. By omitting the intersection with S above, we get $|N_{\le i}(c')| \ge |N_{\le i}(c)| - o(|N_{\le i}(c)|)$ and $|N_{\le i}(c')| \le |N_{\le i}(c)| + o(|N_{\le i+d}(c)|)$.

Now we will show that there exists $q > 0$ such that $A_c(j+1) \le qA_c(j)$ for all j. For a contradiction, suppose that it is not the case. Then, the value of $A_c(j)$ compared to $A_c(j-1)$ at least doubles at infinitely many j's. At these points, $C_c(j) \ge (1/2)A_c(j)$, contradicting $C_c(j) = o(A_c(j))$. Therefore we have $|N_{\le i+d}(c)| \le q^d |N_{\le i}(c)|$ and we can replace $o(|N_{\le i+d}(c)|)$ with $o(|N_{\le i}(c)|)$ in the above expressions.

Applying to the terms of the limit in $\rho_{\mathcal{F}}(G, v, c')$ we get

$$\frac{|N_{\le i}(c') \cap S|}{|N_{\le i}(c')|} \le \frac{|N_{\le i}(c) \cap S| + o(|N_{\le i}(c)|)}{|N_{\le i}(c)| - o(|N_{\le i}(c)|)} \le \frac{|N_{\le i}(c) \cap S|}{|N_{\le i}(c)|} + o(1).$$

Proving the other direction is analogous, and so we get $\rho_{\mathcal{F}}(G, v, c') = \rho_{\mathcal{F}}(G, v, c)$. □

3.1. Infinite square grid.

For the infinite square grid we show that the surviving rate is equal to $1/4$.

Theorem 10. *For the infinite square grid $G_\Box$ we have $\rho(G_\Box) = 1/4$.*

Proof. Let v be the vertex that catches fire initially. Then, after the i-th turn, at least $(3/4)|N_i(v)|$ vertices of $N_i(v)$ burn by Lemma 7. So we have $\rho(G_\Box) \le 1/4$ the by definition of $\rho(G, v)$.

On the other hand, the strategy outlined in Section 2.1, applied to the infinite grid saves vertices in $90°$ wedge, giving $\rho(G_\Box) \ge 1/4$. □

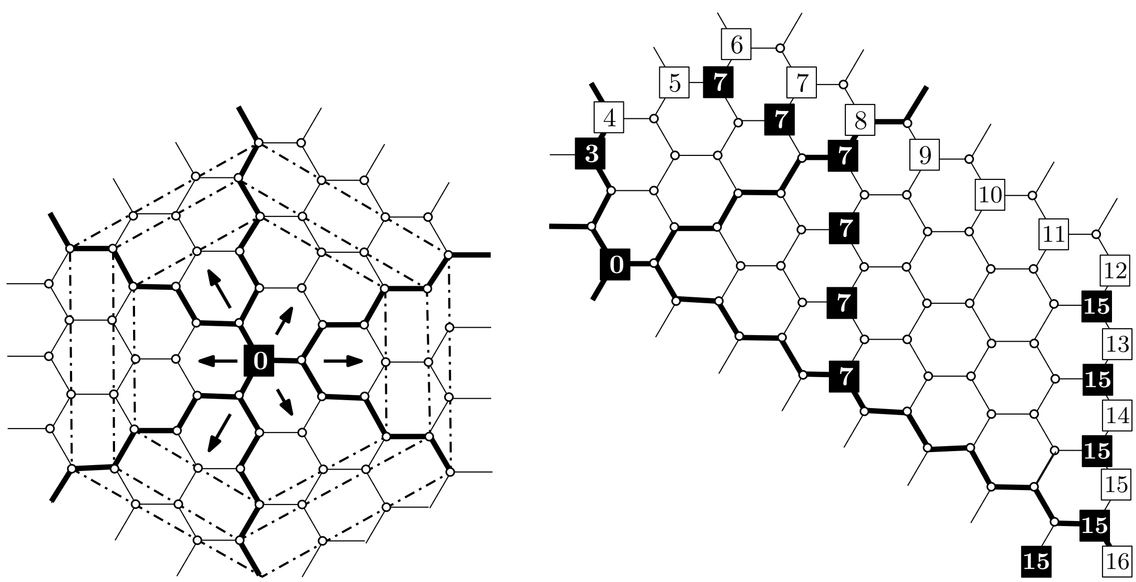
3.2. Infinite hexagonal grid.

When we assume the fire starts at vertex v_0, the hexagonal grid is naturally divided into six $60°$ cones, every other touching v_0. (See Figure 3(a).)

Let us start with the following simple but convenient observation. If there are more burning vertices, we may always take a ball $N_{\le t}(v_0)$ with sufficiently large radius t, and assume that the fire started burning t turns ago at vertex v_0 (and no vertices were saved in the first t turns). We state this as the following observation:

Lemma 11. *For a given graph G and a given firefighter strategy § for the process with $N_{\le t}(v_0)$ burning initially, let F_i be the set of vertices burning after i turns.*

Then, § can be modified to get a strategy §′ that can be used to continue the process in which all active fire A is contained in $N_{\le t}(v_0)$. Let D be the set of vertices with initial inactive fire (which may or may not be outside of $N_{\le t}(v_0)$; there might be some firefighters already introduced, our goal is to continue the process from this initial configuration) and let F'_i be the set of vertices burning after i turns, provide that §′ is applied. In particular, we have $F'_i \subseteq F_i \cup D$.



(a) (b)

FIGURE 3. (a) Six cones of the hex grid centered at v_0, the hexagons indicate $N_4(v_0)$, $N_5(v_0)$ and $N_6(v_0)$. (b) Construction of two segments of the spiral for $t = 4$.

Proof. Let §′ play as § would, except whenever § moves to a vertex that is already burning or protected. In such situation, the firefighter protects an arbitrary vertex. Any vertex in F'_i not in D caught fire via a sequence of adjacent vertices of length at most i starting at some vertex in A. This sequence could not pass through any protected vertex or through an inactive fire (as it never spreads once becomes inactive), so this vertex also catches fire when the strategy § is used for the process starting with $N_{\le t}(v_0)$ (in fact, the sequence might be even shorter). Hence, $F'_i - D \subseteq F_i$, and the result holds. □

Now we show how to build a spiral construction in the hexagonal grid:

Observation 12. *Suppose that, for some v_0 and even $t \ge 1$, the vertices of $N_{\le t-1}(v_0)$ are burning. Then, one firefighter can build a spiral around v_0. The spiral is composed of segments, each segment is contained in one of the six cones. A segment starts on a cone boundary at distance t and ends at the next boundary at distance $2t$. Moreover, this construction can be iterated, see Figure 3(b) for an example of two cones with $t = 4$. Observe that the firefighter placed in move i is in distance at least $i + t - 1$ from v_0.*

Note that the observation can be easily generalized using Lemma 11.

Now we return to and prove the results on G_{hex} stated in Section 1.1.

Proof of Theorem 3. The strategy exploits the fact that in certain directions in the hexagonal grid G_{hex}, we need to protect only one vertex per (graph) distance two along the direction. The firefighter alternates between protecting the vertices of two rays as shown in Figure 4(a). □

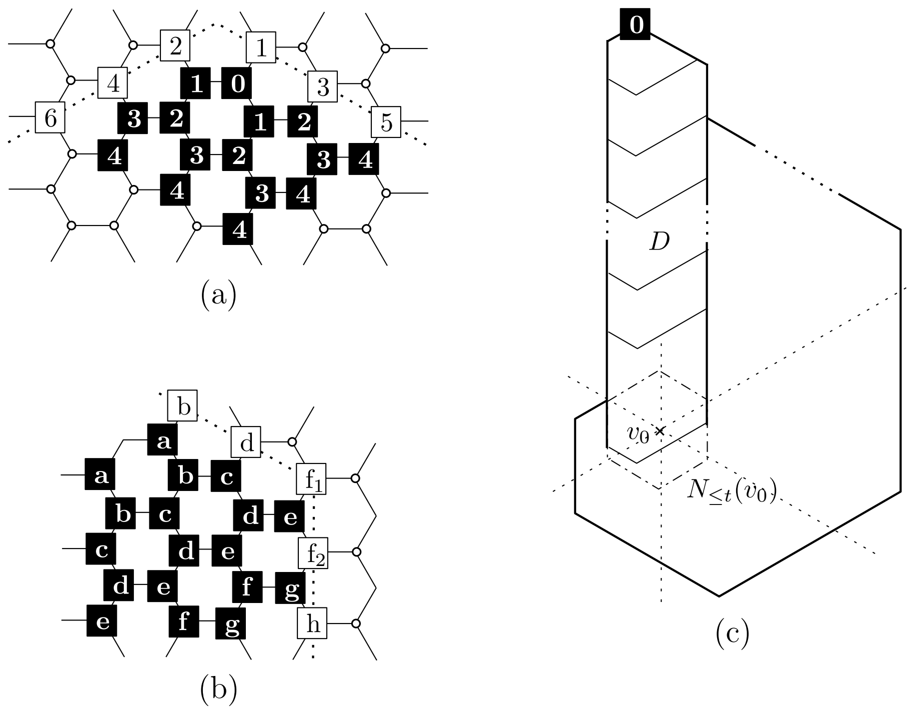


FIGURE 4. (a) Start of the strategy protecting 2/3 of the grid with indicated protected rays. (b) Bending the protected ray with one extra firefighter at time f. (c) Overview of the strategy of Theorem 4. Starting with two bends, we build a long enough strip, then start spiralling. (Note that the spiral is deformed to fit in the figure.) v_0, D and $N_{\leq t}(v_0)$ are as in Observation 12 and Lemma 11, the thin lines indicate the active fire at certain time points.

Proof or Theorem 4. Initially, the strategy is the same as in the proof of Theorem 3 above. When we get the first extra protected vertex, we can bend one of the rays by 60° as indicated in Figure 4(b). The letters indicate consecutive turn numbers, in the turns a, c, e and g, the firefighter protects the other ray, f is the turn at time t_1 with the first extra firefighter. Note that from this point on, 5/6 of the grid is protected—in the limit, the fire would only occupy a 60° wedge.

When we get the second extra firefighter at time t_2, we bend the other ray in a symmetric way. Note that in case the extra firefighter came in a turn we play on the other ray, we can protect the required vertex in advance.

We restricted the fire to a strip extending only in one direction as in Figure 4(c). Note that the strip might be very wide, depending on t_1 and t_2, but the width does not change and there is some t such that in every turn, all the active fire is contained in some ball of radius t. We let the strip grow to length at least $2^7 t$.

Then we abandon protecting the two rays and start building a spiral described in Observation 12 generalizing it with Lemma 11 for our situation. If we start the spiral in the angle indicated in Figure 4(c), the fifth spiral segment hits the wall of the strip and we have enclosed the fire with protected vertices. Since the first segment of the spiral starts in distance t from v_0, the fifth segment ends in distance $2^6 t$ from v_0, so stripe length $2^7 t$ is enough for the spiral and the strip to meet. □

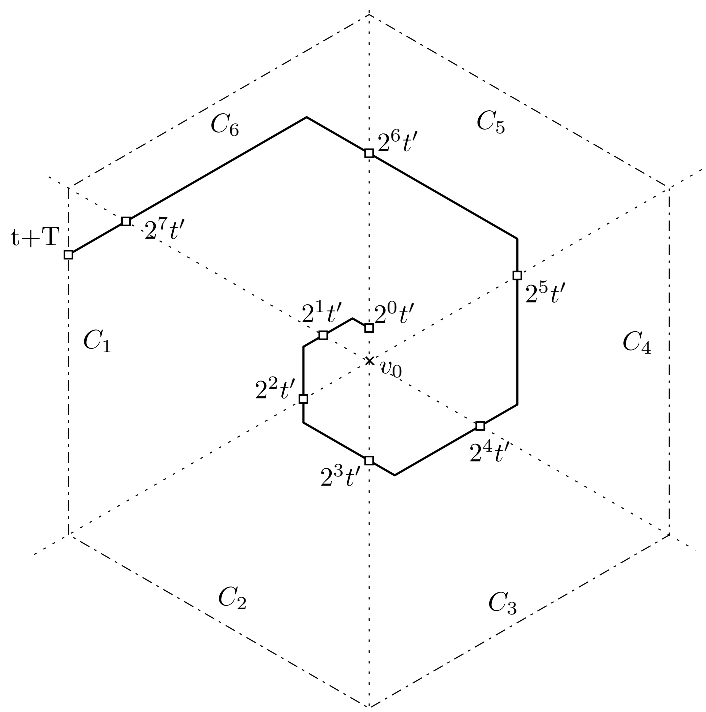


FIGURE 5. Seven consecutive segments of a spiral. The numbers indicate the distances of the segment ends from v_0. (Note that the spiral is deformed to fit the figure.)

Proof of Theorem 5. Choose v_0 and t so that all burning vertices are contained in $N_{\leq t-1}(v_0)$. We use Lemma 11 and in the rest of the proof assume that exactly $N_{\leq t-1}(v_0)$ is burning initially, and there are no protected vertices. Our strategy is then to indefinitely build a spiral, as described in Observation 12, starting at distance t.

Now consider turn T and assume $T \geq t^8$. Note that the spiral is always built to at least touch $N_{T+t}(v_0)$. Let t' be such that $t' = 2^k t$ for some k and $2^7 t' \leq T + t < 2^8 t'$. The situation with seven largest segments is illustrated on Figure 5.

Informally, we notice that even when considering only the last seven consecutive segments of the spiral, the fire has to take a detour proportional to the size of the smallest considered segment to reach any vertex of $N_{T+t}(v_0)$.

More precisely, let $C_1, C_2, \ldots, C_6$ denote the cones around v_0. Let $B_i = N_{T+t}(v_0) \cap C_i$. To reach B_i from v_0, the fire has to travel at least the distance $v_0 \ldots 2^{i-1}t' \ldots B_i$. Any such path has length at least $2^{i-1}t' + (T + t)$ which is minimal for B_1. The actual graph distance $v_0 \ldots B_i$ is $T + t$.

Therefore, we can take any constant c such that for T large enough we have

$$c > \frac{T+t}{2^0 t' + (T+t)}.$$

This is satisfied by $c > \frac{1}{1+2^{-8}} \approx 0.996$ for T large enough compared to t, which can be ensured by the choice of n_0. □

References

[1] L. Cai, Y. Cheng, E. Verbin, Y. Zhou, Surviving Rates of Graphs with Bounded Treewidth for the Firefighter Problem, *SIAM Journal on Discrete Mathematics* **24(4)** (2010), 1322–1335.

[2] L. Cai and W. Wang, The surviving rate of a graph, *SIAM Journal of Discrete Mathematics* **23(4)** (2009), 1814–1826.

[3] O.N. Feldheim and R. Hod, 3/2 firefighters are not enough, *Discrete Applied Mathematics* **161** (2013), 301–306.

[4] S. Finbow, B. Hartnell, Q. Li, K. Schmeisser, On minimizing the effects of fire or a virus on a network, Papers in honour of Ernest J. Cockayne, *J. Combin. Math. Combin. Comput.* **33** (2000), 311–322.

[5] S. Finbow, A. King, G. MacGillivray, and R. Rizzi, The firefighter problem for graphs of maximum degree three, *Discrete Mathematics* **307** (2007), 2094–2105.

[6] S. Finbow and G. MacGillivray, The firefighter problem: a survey of results, directions and questions. *Australasian Journal of Combinatorics* **43** (2009), 57–77.

[7] S. Finbow, P. Wang, and W. Wang, The surviving rate of an infected network, *Theoretical Computer Science* **411** (2010) 3651–3660.

[8] P. Fogarty, Catching the Fire on Grids, M.Sc. Thesis, University of Vermont (2003).

[9] B. Hartnell, Firefighter! An application of domination. Presentation at the 25th Manitoba Conference on Combinatorial Mathematics and Computing, University of Manitoba, Winnipeg, Canada, 1995.

[10] B. Hartnell and Q. Li, Firefighting on trees: how bad is the greedy algorithm?, *Congr Numer* **145** (2000), 187–192.

[11] P. Prałat, Graphs with average degree smaller than $\frac{30}{11}$ burn slowly, to appear in *Graphs and Combinatorics*.

[12] P. Prałat, Sparse graphs are not flammable, preprint.

[13] P. Wang and S.A. Moeller, Fire Control on Graphs, *J. Combin. Math. Combin. Comput.* **41** (2000), 19–34.

[14] Maple worksheet available on-line: `http://www.math.ryerson.ca/~pralat/`

Appendix A. Proof of the upper bound from Theorem 1

Here we examine the upper bound in the general case when a fire breaks out at a vertex (a, b) of $P_n \Box P_n$. As before, due to the symmetry, we may assume that $a = xn + o(1)$, $b = yn + o(1)$ for some $0 \le x \le 1/2$ and $0 \le y \le x$. We want to investigate the size of $N_r(a, b)$ so we are interested in two things: the time $(t_i + o(1))n$ when we reach each of the 4 borders $(i = N, S, E, W)$, and the time $(t_i + o(1))n$ when each of the 4 fire fronts $(i = NE, SE, SW, NW)$ disappear. Clearly,

$$t_E = \frac{1}{2} - x, \quad t_N = \frac{1}{2} - y, \quad t_S = \frac{1}{2} + y, \quad t_W = \frac{1}{2} + x,$$

and $t_E \le t_N \le t_S \le t_W$. Moreover,

$$t_{NE} = 1 - x - y, \quad t_{SE} = 1 - x + y, \quad t_{NW} = 1 + x - y, \quad t_{SW} = 1 + x + y,$$

and $t_{NE} \le t_{SE} \le t_{NW} \le t_{SW}$. The order in which these events occur determines the formula for $|N_r(a, b)|$. It is easy to see that $t_{NE} \ge t_N$, $t_{SE} \ge t_S$, and $t_{NW} \ge t_W$, and so there are only 5 cases to consider. Unfortunately, since the number of vertices burnt is at least $\sum_{r \ge 1} \max(|N_r(0, 0)| - r, 0)$ (by Lemma 7), sometimes we need to consider some sub-cases depending on in which time interval $|N_r(0, 0)| - r$ becomes negative. Let $(t + o(1))n$ be the first time this happens. The calculations are elementary but quite tedious, so we refer the reader to the Maple worksheet [14] to check integrals, etc.

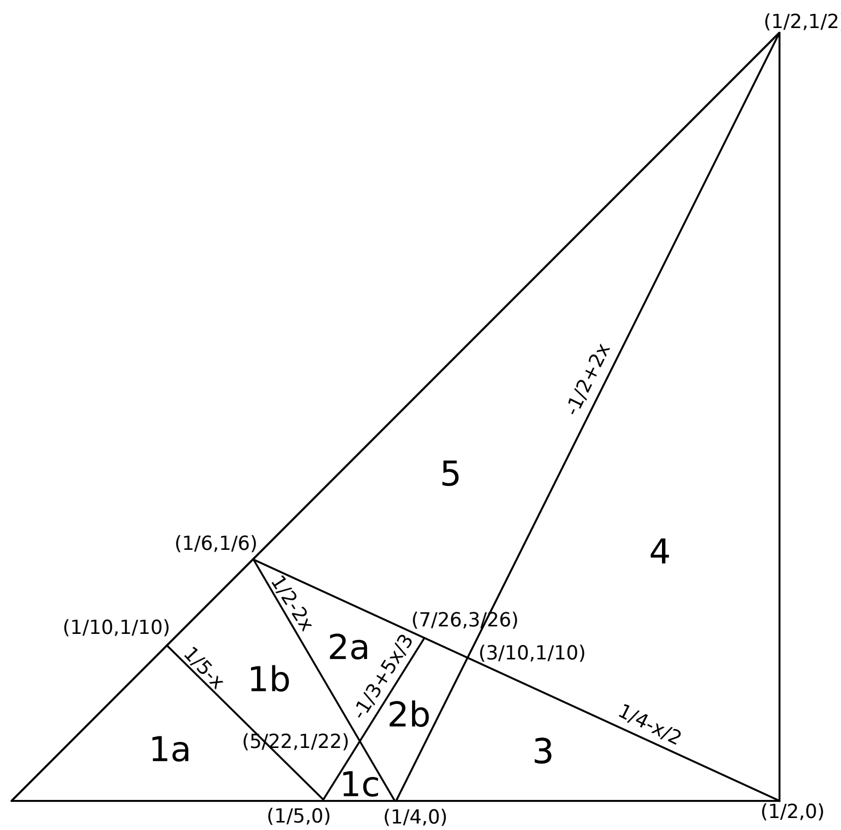


Figure 6. The regions corresponding to the individual cases.

Case 1: $t_E \leq t_N \leq t_S \leq t_W \leq t_{NE} \leq t_{SE} \leq t_{NW} \leq t_{SW}$.
The condition $t_W \leq t_{NE}$ is equivalent to $y \leq 1/2 - 2x$ so we are concerned with the Region 1 presented on Figure 6. The number of vertices at distance rn from (a,b) (for some $r = r(n)$) behaves as follows:

$$\frac{|N_{rn}(a,b)|}{n} = o(1) + \begin{cases} 4r & \text{if } 0 \leq r \leq t_E \\ 1-2x+2r & \text{if } t_E \leq r \leq t_N \\ 2-2x-2y & \text{if } t_N \leq r \leq t_S \\ 3-2x-2r & \text{if } t_S \leq r \leq t_W \\ 4-4r & \text{if } t_W \leq r \leq t_{NE} \\ 3+x+y-3r & \text{if } t_{NE} \leq r \leq t_{SE} \\ 2+2x-2r & \text{if } t_{SE} \leq r \leq t_{NW} \\ 1+x+y-r & \text{if } t_{NW} \leq r \leq t_{SW}. \end{cases}$$

Case 1a: $t_W \leq t \leq t_{NE}$.
The condition $t \leq t_{NE}$ implies that $y \leq 1/5 - x$ (as before, we direct the reader to Figure 6). Provided (a,b) is from the region we consider in this sub-case, $t = 4/5$ (note that $|N_{rn}(a,b)| - rn = (4 - 5r + o(1))n$ for $t_W \leq r \leq t_{NE}$). Hence, the proportion of vertices burnt is at least

$$\begin{aligned} B_{1a}(x,y) \quad := \quad & \int_0^{t_E} 3r dr + \int_{t_E}^{t_N} (1-2x+r)dr + \int_{t_N}^{t_S} (2-2x-2y-r)dr \\ & + \int_{t_S}^{t_W} (3-2x-3r)dr + \int_{t_W}^{t} (4-5r)dr = \frac{3}{5} - 2x^2 - 2y^2. \end{aligned}$$

The contribution from vertices from this region is calculated as follows (as usual, we refer to Figure 6).

$$C_{1a} := \int_0^{1/10} \int_0^x B_{1a}(x,y)dydx + \int_{1/10}^{1/5} \int_0^{1/5-x} B_{1a}(x,y)dydx = \frac{43}{7500} \approx 0.005733.$$

Case 1b: $t_{NE} \leq t \leq t_{SE}$.
The condition $t \leq t_{SE}$ implies that $y \geq -1/3 + 5x/3$. It follows that $t = 3/4 + x/4 + y/4$ and the proportion of vertices burnt is at least

$$\begin{aligned} B_{1b}(x,y) \quad := \quad & \int_0^{t_E} 3r dr + \int_{t_E}^{t_N} (1-2x+r)dr + \int_{t_N}^{t_S} (2-2x-2y-r)dr \\ & + \int_{t_S}^{t_W} (3-2x-3r)dr + \int_{t_W}^{t_{NE}} (4-5r)dr + \int_{t_{NE}}^{t} (3+x+y-4r)dr \\ = \quad & \frac{5}{8} - \frac{x}{4} - \frac{11x^2}{8} - \frac{y}{4} - \frac{11y^2}{8} + \frac{5xy}{4}. \end{aligned}$$

The contribution from vertices from this region is

$$\begin{aligned} C_{1b} \quad := \quad & \int_{1/10}^{1/6} \int_{1/5-x}^{x} B_{1b}(x,y)dydx + \int_{1/6}^{1/5} \int_{1/5-x}^{1/2-2x} B_{1b}(x,y)dydx \\ & + \int_{1/5}^{5/22} \int_{-1/3+5x/3}^{1/2-2x} B_{1b}(x,y)dydx = \frac{459563}{89842500} \approx 0.005115. \end{aligned}$$

Case 1c: $t_{SE} \le t \le t_{NW}$**.**
The condition $t \le t_{NW}$ is equivalent to $y \le 1/3 + x/3$ which is satisfied by all points in the Case 1. It follows that $t = 2/3 + 2x/3$ and the proportion of vertices burnt is at least

$$\begin{aligned} B_{1c}(x,y) &:= \int_0^{t_E} 3r dr + \int_{t_E}^{t_N} (1-2x+r)dr + \int_{t_N}^{t_S} (2-2x-2y-r)dr \\ &\quad + \int_{t_S}^{t_W} (3-2x-3r)dr + \int_{t_W}^{t_{NE}} (4-5r)dr + \int_{t_{NE}}^{t_{SE}} (3+x+y-4r)dr \\ &\quad + \int_{t_{SE}}^{t} (2+2x-3r)dr = \frac{2}{3} - \frac{2x}{3} - \frac{x^2}{3} - y^2. \end{aligned}$$

The contribution from vertices from this region is

$$\begin{aligned} C_{1c} &:= \int_{1/5}^{5/22} \int_0^{-1/3+5x/3} B_{1c}(x,y) dy dx + \int_{5/22}^{1/4} \int_0^{1/2-2x} B_{1c}(x,y) dy dx \\ &= \frac{434549}{766656000} \approx 0.000567. \end{aligned}$$

Case 2: $t_E \le t_N \le t_S \le t_{NE} \le t_W \le t_{SE} \le t_{NW} \le t_{SW}$**.**
The condition $t_S \le t_{NE}$ is equivalent to $y \le 1/4 - x/2$ and the condition $t_W \le t_{SE}$ to $y \ge -1/2 + 2x$, so we are concerned with the Region 2 presented on Figure 6. This time

$$\frac{|N_{rn}(a,b)|}{n} = o(1) + \begin{cases} 4r & \text{if } 0 \le r \le t_E \\ 1-2x+2r & \text{if } t_E \le r \le t_N \\ 2-2x-2y & \text{if } t_N \le r \le t_S \\ 3-2x-2r & \text{if } t_S \le r \le t_{NE} \\ 2-x+y-r & \text{if } t_{NE} \le r \le t_W \\ 3+x+y-3r & \text{if } t_W \le r \le t_{SE} \\ 2+2x-2r & \text{if } t_{SE} \le r \le t_{NW} \\ 1+x+y-r & \text{if } t_{NW} \le r \le t_{SW}. \end{cases}$$

Case 2a: $t_W \le t \le t_{SE}$**.**
The condition $t \le t_{SE}$ implies that $y \ge -1/3 + 5x/3$. It follows that $t = 3/4 + x/4 + y/4$ and the proportion of vertices burnt is at least

$$\begin{aligned} B_{2a}(x,y) &:= \int_0^{t_E} 3r dr + \int_{t_E}^{t_N} (1-2x+r)dr + \int_{t_N}^{t_S} (2-2x-2y-r)dr \\ &\quad + \int_{t_S}^{t_{NE}} (3-2x-3r)dr + \int_{t_{NE}}^{t_W} (2-x+y-2r)dr \\ &\quad + \int_{t_W}^{t} (3+x+y-4r)dr \\ &= \frac{5}{8} - \frac{x}{4} - \frac{11x^2}{8} - \frac{y}{4} - \frac{11y^2}{8} + \frac{5xy}{4} = B_{1b}(x,y). \end{aligned}$$

The contribution from vertices from this region is

$$\begin{aligned} C_{2a} &:= \int_{1/6}^{5/22}\int_{1/2-2x}^{1/4-x/2} B_{2a}(x,y)dydx + \int_{5/22}^{7/26}\int_{-1/3+5x/3}^{1/4-x/2} B_{2a}(x,y)dydx \\ &= \frac{358687}{157907178} \approx 0.002275. \end{aligned}$$

Case 2b: $t_{SE} \le t \le t_{NW}$**.**
The condition $t \le t_{NW}$ is equivalent to $y \le 1/3 + x/3$ which is satisfied by all points in the Case 2. It follows that $t = 2/3 + 2x/3$ and the proportion of vertices burnt is at least

$$\begin{aligned} B_{2b}(x,y) &:= \int_0^{t_E} 3rdr + \int_{t_E}^{t_N}(1-2x+r)dr + \int_{t_N}^{t_S}(2-2x-2y-r)dr \\ &\quad + \int_{t_S}^{t_{NE}}(3-2x-3r)dr + \int_{t_{NE}}^{t_W}(2-x+y-2r)dr \\ &\quad + \int_{t_W}^{t_{SE}}(3+x+y-4r)dr + \int_{t_{SE}}^{t}(2+2x-3t)dr \\ &= \frac{2}{3} - \frac{2x}{3} - \frac{x^2}{3} - y^2 = B_{1c}(x,y). \end{aligned}$$

The contribution from vertices from this region is

$$\begin{aligned} C_{2b} &:= \int_{5/22}^{1/4}\int_{1/2-2x}^{-1/3+5x/3} B_{2b}(x,y)dydx + \int_{1/4}^{7/26}\int_{-1/2+2x}^{-1/3+5x/3} B_{2b}(x,y)dydx \\ &\quad + \int_{7/26}^{3/10}\int_{-1/2+2x}^{1/4-x/2} B_{2b}(x,y)dydx = \frac{478988221}{280723872000} \approx 0.001707. \end{aligned}$$

Case 3: $t_E \le t_N \le t_S \le t_{NE} \le t_{SE} \le t_W \le t_{NW} \le t_{SW}$**.**
In this case, we are concerned with the Region 3 presented on Figure 6. This time

$$\frac{|N_{rn}(a,b)|}{n} = o(1) + \begin{cases} 4r & \text{if } 0 \le r \le t_E \\ 1-2x+2r & \text{if } t_E \le r \le t_N \\ 2-2x-2y & \text{if } t_N \le r \le t_S \\ 3-2x-2r & \text{if } t_S \le r \le t_{NE} \\ 2-x+y-r & \text{if } t_{NE} \le r \le t_{SE} \\ 1 & \text{if } t_{SE} \le r \le t_W \\ 2+2x-2r & \text{if } t_W \le r \le t_{NW} \\ 1+x+y-r & \text{if } t_{NW} \le r \le t_{SW}. \end{cases}$$

In this case, $t_W \le t \le t_{NW}$. It follows that $t = 2/3 + 2x/3$ and the proportion of vertices burnt is at least

$$\begin{aligned} B_3(x,y) &:= \int_0^{t_E} 3r dr + \int_{t_E}^{t_N} (1-2x+r)dr + \int_{t_N}^{t_S} (2-2x-2y-r)dr \\ &\quad + \int_{t_S}^{t_{NE}} (3-2x-3r)dr + \int_{t_{NE}}^{t_{SE}} (2-x+y-2r)dr \\ &\quad + \int_{t_{SE}}^{t_W} (1-r)dr + \int_{t_W}^{t} (2+2x-3t)dr \\ &= \frac{2}{3} - \frac{2x}{3} - \frac{x^2}{3} - y^2 = B_{2b}(x,y) = B_{1c}(x,y). \end{aligned}$$

The contribution from vertices from this region is

$$C_3 := \int_{1/4}^{3/10} \int_0^{-1/2+2x} B_3(x,y)dydx + \int_{3/10}^{1/2} \int_0^{1/4-x/2} B_3(x,y)dydx = \frac{2807}{576000} \approx 0.004873.$$

Case 4: $t_E \le t_N \le t_{NE} \le t_S \le t_{SE} \le t_W \le t_{NW} \le t_{SW}$**.**
In this case, we are concerned with the Region 4 presented on Figure 6. This time

$$\frac{|N_{rn}(a,b)|}{n} = o(1) + \begin{cases} 4r & \text{if } 0 \le r \le t_E \\ 1-2x+2r & \text{if } t_E \le r \le t_N \\ 2-2x-2y & \text{if } t_N \le r \le t_{NE} \\ 1-x-y+r & \text{if } t_{NE} \le r \le t_S \\ 2-x+y-r & \text{if } t_S \le r \le t_{SE} \\ 1 & \text{if } t_{SE} \le r \le t_W \\ 2+2x-2r & \text{if } t_W \le r \le t_{NW} \\ 1+x+y-r & \text{if } t_{NW} \le r \le t_{SW}. \end{cases}$$

In this case, $t_W \le t \le t_{NW}$. It follows that $t = 2/3 + 2x/3$ and the proportion of vertices burnt is at least

$$\begin{aligned} B_4(x,y) &:= \int_0^{t_E} 3r dr + \int_{t_E}^{t_N} (1-2x+r)dr + \int_{t_N}^{t_{NE}} (2-2x-2y-r)dr \\ &\quad + \int_{t_{NE}}^{t_S} (1-x-y)dr + \int_{t_S}^{t_{SE}} (2-x+y-2r)dr \\ &\quad + \int_{t_{SE}}^{t_W} (1-r)dr + \int_{t_W}^{t} (2+2x-3t)dr \\ &= \frac{2}{3} - \frac{2x}{3} - \frac{x^2}{3} - y^2 = B_3(x,y) = B_{2b}(x,y) = B_{1c}(x,y). \end{aligned}$$

The contribution from vertices from this region is

$$C_4 := \int_{3/10}^{1/2} \int_{1/4-x/2}^{-1/2+2x} B_4(x,y)dydx = \frac{473}{36000} \approx 0.013139.$$

Case 5: $t_E \leq t_N \leq t_{NE} \leq t_S \leq t_W \leq t_{SE} \leq t_{NW} \leq t_{SW}$**.**
In this case, we are concerned with the Region 5 presented on Figure 6. This time

$$\frac{|N_{rn}(a,b)|}{n} = o(1) + \begin{cases} 4r & \text{if } 0 \leq r \leq t_E \\ 1-2x+2r & \text{if } t_E \leq r \leq t_N \\ 2-2x-2y & \text{if } t_N \leq r \leq t_{NE} \\ 1-x-y+r & \text{if } t_{NE} \leq r \leq t_S \\ 2-x+y-r & \text{if } t_S \leq r \leq t_W \\ 3+x+y-3r & \text{if } t_W \leq r \leq t_{SE} \\ 2+2x-2r & \text{if } t_{SE} \leq r \leq t_{NW} \\ 1+x+y-r & \text{if } t_{NW} \leq r \leq t_{SW}. \end{cases}$$

In this case, $t_S \leq t \leq t_W$. It follows that $t = 1 - x/2 + y/2$ and the proportion of vertices burnt is at least

$$\begin{aligned} B_5(x,y) &:= \int_0^{t_E} 3r dr + \int_{t_E}^{t_N} (1-2x+r)dr + \int_{t_N}^{t_{NE}} (2-2x-2y-r)dr \\ &\quad + \int_{t_{NE}}^{t_S} (1-x-y)dr + \int_{t_S}^{t} (2-x+y-2r)dr \\ &= \frac{3}{4} - x - \frac{x^2}{4} - \frac{5y^2}{4} + \frac{xy}{2}. \end{aligned}$$

The contribution from vertices from this region is

$$C_5 := \int_{1/6}^{3/10} \int_{1/4-x/2}^{x} B_5(x,y)dydx + \int_{3/10}^{1/2} \int_{-1/2+2x}^{x} B_5(x,y)dydx = \frac{1907}{162000} \approx 0.638585.$$

Finally, since there are 8 symmetric regions to consider,

$$\begin{aligned} \rho(P_n \square P_n) &\leq 1 - 8(C_{1a} + C_{1b} + C_{1c} + C_{2a} + C_{2b} + C_3 + C_4 + C_5) + o(1) \\ &= \frac{67243}{105300} + o(1) < 0.6386. \end{aligned}$$

The proof of the upper bound is finished.

Department of Applied Mathematics, Charles University, Praha, Czech Republic
E-mail address: gavento@kam.mff.cuni.cz

Department of Applied Mathematics, Charles University, Praha, Czech Republic
E-mail address: honza@kam.mff.cuni.cz

Department of Mathematics, Ryerson University, Toronto, ON, Canada
E-mail address: pralat@ryerson.ca